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Control Barrier Functionals: Safety-critical Control for Time Delay Systems

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Summary

This work presents a theoretical framework for the safety-critical control of time delay systems. The theory of control barrier functions, that provides formal safety guarantees for delay-free systems, is extended to systems with state delay. The notion of control barrier functionals is introduced, to attain formal safety guarantees by enforcing the forward invariance of safe sets defined in the infinite dimensional state space. The proposed framework is able to handle multiple delays and distributed delays both in the dynamics and in the safety condition, and provides an affine constraint on the control input that yields provable safety. This constraint can be incorporated into optimization problems to synthesize pointwise optimal and provable safe controllers. The applicability of the proposed method is demonstrated by numerical simulation examples.

KEYWORDS:

Control of nonlinear systems, Safety-critical control, Delay systems, Infinite dimensional systems

1 | INTRODUCTION

Modern control systems place high priority on safety, which is frequently a precursor to other control goals such as performance, efficiency, and sustainability. There exist numerous examples—from self-driving autonomous vehicles[1], through robotic systems[2,3,4] to human-robot collaboration[5,6,7]—where safety plays a key role for reliable autonomy or sustainable operation. Safety is crucial even outside engineering, including biological applications and epidemiological models that describe pandemics[8,9]. Therefore, establishing safety-critical control techniques is of high significance with wide-spread application domains.

To formally address safety in dynamical and control systems, one can define a safe set over the state space, and safety can be framed as the forward invariance of that set: the system must evolve within the safe set for all time. Rigorous guarantees of safety necessitate a theory for ensuring forward set invariance. Barrier functions (or safety functions) have been established to certify set invariance in dynamical systems[10,11,12,13,14], while control barrier functions (CBFs) enable safe controller synthesis in control systems. The framework of CBFs was first introduced in[15] and later refined in[16]. A comprehensive review of safety-critical control can be found in[17] and the references therein.

While most works in safety-critical control are applied to delay-free systems, time delays often occur in many applications. For example, the reflex delay of human operators affects human-machine interactions; models of vehicular traffic include the reaction time of drivers as delays[18]; wheel-shimmy motion – experienced on vehicles due to the elastic contact between tires and the road – can be captured using models with distributed delay[19]; manufacturing processes like metal cutting may suffer from vibrations

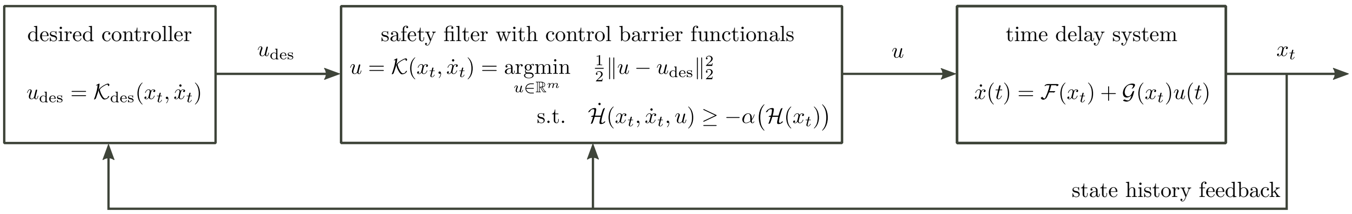

FIGURE 1 Block diagram of the proposed control framework that provides formal safety guarantees for the control loops of time delay systems. A safety filter, that is established based on the proposed *control barrier functionals*, is applied to modify a nominal (not necessarily safe) control input and guarantee safe behavior for time delay systems through state history feedback.

due to the delayed regenerative effect of chip formation [20]; hydraulic systems include time delays caused by wave propagation in pipes [21,22]; and epidemiological models contain delays due to the incubation period of infectious diseases [23,24]. Time delay also plays important role in population dynamics [25], neural networks [26], brain dynamics [27], human sensory system [28,29], and robotic systems [30]. Such time delays may render control systems unsafe if controllers are designed without considering the delay.

Generally speaking, delays can enter a control system in two different ways: in the control input or in the state. When a delay appears in the control input, the dynamics are often formulated as:

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t - \tau). \tag{1}$$

With the input delay τ, the input affects the system after a delay period, hence it needs to be taken into account what will happen to the system in the future before the input becomes effective. Therefore, the idea of predictor feedback [31,32,33,34] is often used to eliminate the effect of the delay by predicting the future state from the actual state and the input history. Comprehensive literature about safety-critical control with input delay for various applications can be found in [35,36] for discrete time and in [37,8,9,38,39] for continuous time, while the works in [40,41] include time-varying and multiple input delays.

When a delay appears in the state, a typical example for control systems becomes:

$$\dot{x}(t) = f\big(x(t), x(t - \tau)\big) + g\big(x(t), x(t - \tau)\big)u(t). \tag{2}$$

For the case of state delay, safety-critical control has not yet been fully addressed to the best of our knowledge. Therefore, this paper intends to tackle this problem. We seek to find controllers for time delay systems like (2) such that safety is maintained. Specifically, we seek to keep a certain scalar safety measure h positive, wherein the safety condition may also depend on delayed states (as it was first proposed in [42]). For example, we require the following to hold for the system to be considered safe:

$$h\big(x(t), x(t - \tau)\big) \geq 0, \quad \forall t \geq 0. \tag{3}$$

The main challenge of controlling systems with state delay originates from the infinite dimensional nature of delayed dynamics. Namely, the state of the system is a function over the delay period, that implies an infinite dimensional state space. As such, time delay systems are often described by functional differential equations (FDEs) [43,44,45,46,47], while scalar safety measures can be constructed as functionals of the state. There exist a few instances of using functionals in the literature in the context of safety. Safety verification for partial differential equations using barrier functionals can be found in [48] and state-constrained control considering integral barrier Lyapunov functionals are discussed in [49,50]. For autonomous time delay systems without control, [51] introduced the concept of safety functionals, which has been investigated further in [52] by means of discretization. The relationship between discretization and functionals is discussed in [53], while safe domains interpreted as basin of attraction were investigated for delayed dynamical systems in [54]. These works, however, do not address control systems with time delay.

1.1 | Contributions

The main contribution of this work is the establishment of a framework for control synthesis with formal safety guarantees for control systems with state delay. While this includes, for instance, systems of the form (2) with safety requirement (3), we discuss a much broader class of time delay systems and safety conditions. To achieve safety, we introduce the concept of *control barrier functionals* as a tool for synthesizing safety-critical controllers, through building on the existing notions of safety functionals [51] and control barrier functions [16]. The corresponding controllers can be constructed as safety filters illustrated in Fig. 1. We use the theory of retarded and neutral functional differential equations to prove the underlying formal safety guarantees.

We remark that a few recent papers have also approached this problem parallel to our work. Namely,[55] considers delays with disturbances while[56,57,58] investigate the combination of stability and safety by the application of Razumikhin- and Krasovskii-type control Lyapunov and control barrier functionals. Although these recent works share some of the ideas presented in this paper, we establish a comprehensive in-depth study that is not covered by previous works. This includes a wider class of control barrier functionals, an exhaustive discussion on how to calculate the derivatives of these functionals, the safety of neutral FDEs, the notion of relative degree for time delay systems, and numerous application examples. However, we do not address questions related to stability or disturbances.

The rest of the paper is organized as follows. In Section 2, safety is revisited for delay-free dynamical and control systems through the notions of safety functions and control barrier functions, respectively. Then, Sections 3 and 4 present the major contributions of this work: formal guarantees of safety for time delay systems. Section 3 establishes the theoretical foundations of safety functionals that certify the safety of autonomous delayed dynamical systems, while Section 4 discusses safety-critical control with state delay by means of control barrier functionals. In Section 5, we demonstrate safety-critical control on illustrative examples and a more practical case study through the regulated delayed predator-prey problem. Finally, we conclude our results and discuss future research directions in Section 6.

2 | SAFETY OF DELAY-FREE SYSTEMS

In this section, we revisit safety certification for delay-free dynamical systems and safety-critical control for delay-free control systems, that are described by ordinary differential equations (ODEs). Specifically, we focus on the notions of *safety functions* and *control barrier functions*. Then, in Section 3, we extend these frameworks to time delay systems.

2.1 | Dynamical Systems

Consider the dynamical system described by the ODE:

$$\dot{x}(t) = f\big(x(t)\big), \tag{4}$$

where dot represents derivative with respect to time t, $x \in \mathbb{R}^n$ is the state variable, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous function. Given an initial condition $x(0) \in \mathbb{R}^n$, this system has a unique solution over $t \in I(x(0))$ with an interval of existence $I(x(0)) \subseteq \mathbb{R}$. For simplicity of exposition, throughout this paper we assume $I(x(0)) = \mathbb{R}_{\geq 0}$, i.e., solutions exist $\forall t \geq 0$.

We consider system (4) safe when the solution $x(t)$ evolves within a *safe set* $S \subset \mathbb{R}^n$, as given by the following definition.

Definition 1 (**Safety and Forward Invariance**). System (4) is safe w.r.t. set $S \subset \mathbb{R}^n$, if S is forward invariant w.r.t. (4) such that $x(0) \in S \implies x(t) \in S, \forall t \geq 0$ for the solution of (4).

Specifically, we consider set S to be the 0-superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, that is:

$$S = \{x \in \mathbb{R}^n : h(x) \geq 0\}. \tag{5}$$

The forward invariance of set S, therefore, implies that if the initial condition $x(0)$ is within the safe set and satisfies $h(x(0)) \geq 0$, then the solution $x(t)$ stays in the safe set for all time and satisfies $h(x(t)) \geq 0$, $\forall t \geq 0$. For safe sets defined by (5), safety functions allow us to certify the safety of (4).

Definition 2 (**Safety Function**). A continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a **safety function** for (4) on S defined by (5) if there exists $\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$ (see footnote[1]) such that $\forall x \in \mathbb{R}^n$:

$$\dot{h}(x) \geq -\alpha\big(h(x)\big), \tag{6}$$

where $\dot{h}(x) = \nabla h(x) f(x) = L_f h(x)$ is the derivative of h along (4) that is equal to the Lie derivative $L_f h$ of h along f. Here $\nabla h(x) \in \mathbb{R}^{1\times n}$ is a row vector while $f(x) \in \mathbb{R}^{n\times 1}$ is a column vector, and $\nabla h(x) f(x)$ denotes the scalar product of these vectors.

Further technical details with discussion about α can be found in[59]. With this definition, the main result of[16] establishes the safety of dynamical systems.

Theorem 1. [16] *Set S in* (5) *is forward invariant w.r.t.* (4) *if h is a safety function for* (4) *on S, i.e.,* (6) *is satisfied.*

[1] Function $\alpha : \mathbb{R} \to \mathbb{R}$ is of extended class-$\mathcal{K}_\infty$, denoted as $\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$, if α is continuous, monotonically increasing, $\alpha(0) = 0$ and $\lim_{r\to\pm\infty} \alpha(r) = \pm\infty$.

The proof can be found in Appendix A and will be used as basic idea to establish safety for systems with time delay.

Safety functions and Theorem 1 provide a useful tool for certifying the safety of dynamical systems: one needs to verify that (6) holds. A similar concept can be used in control systems to design controllers that enforce the safety of the closed-loop dynamics, which is discussed next.

2.2 | Safety-critical Control

Now consider the affine control system:

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t), \tag{7}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, and locally Lipschitz continuous functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^{n\times m} \to \mathbb{R}^n$. We seek to find a locally Lipschitz continuous controller $k : \mathbb{R}^n \to \mathbb{R}^m$, $u = k(x)$ to enforce that the closed-loop system:

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)k\big(x(t)\big), \tag{8}$$

is safe w.r.t. the set S in (5), that is, S is forward invariant w.r.t. (8). Note that the local Lipschitz continuity of f, g and k ensures that (8) has a unique solution $x(t)$ for any initial condition $x(0) \in \mathbb{R}^n$, and for simplicity we assume that the interval of existence is $t \geq 0$. The safety requirement motivates the following definition.

Definition 3 (Control Barrier Function, CBF[16]). A continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a **control barrier function** (CBF) (see footnote[2]) for (7) on S defined by (5), if there exists $\alpha \in \mathcal{K}^{\rm e}_\infty$ such that $\forall x \in \mathbb{R}^n$:

$$\sup_{u\in\mathbb{R}^m} \big[\dot{h}(x,u)\big] > -\alpha\big(h(x)\big), \tag{9}$$

where

$$\dot{h}(x,u) = \underbrace{\nabla h(x) f(x)}_{L_f h(x)} + \underbrace{\nabla h(x) g(x)}_{L_g h(x)}\, u \tag{10}$$

is the derivative of h along (7), given by the Lie derivatives $L_f h$ and $L_g h$ of h along f and g.

With this definition, control systems can be rendered safe with the following extension of Theorem 1.

Theorem 2. [16] *If h is a CBF for* (7) *on S defined by* (5)*, then any locally Lipschitz continuous controller $k : \mathbb{R}^n \to \mathbb{R}^m$, $u = k(x)$ satisfying:*

$$\dot{h}\big(x, k(x)\big) \geq -\alpha\big(h(x)\big) \tag{11}$$

$\forall x \in S$ renders set S forward invariant w.r.t. (8).

Proof[16]. Definition 3 ensures that controller k exists. Then, considering the closed-loop dynamics in (8), the set S is forward invariant according to Theorem 1. □

This result provides systematic means to safety-critical controller synthesis: one needs to satisfy condition (11) when designing the controller. Condition (11) can be incorporated into optimization problems as constraint to find pointwise optimal safety-critical controllers. For example, one can modify a desired controller $k_{\rm des}$ in a minimally invasive fashion to a safe controller k by solving a quadratic program, as stated formally below.

Corollary 1. *Given a CBF h and a locally Lipschitz continuous desired controller $k_{\rm des} : \mathbb{R}^n \to \mathbb{R}^m$, $u_{\rm des} = k_{\rm des}(x)$, the following quadratic program (QP) yields a controller $k : \mathbb{R}^n \to \mathbb{R}^m$, $u = k(x)$ that renders set S in* (5) *forward invariant w.r.t.* (8)*:*

$$\begin{aligned} k(x) = \underset{u\in\mathbb{R}^m}{\operatorname{argmin}} \quad & \frac{1}{2}\|u - k_{\rm des}(x)\|_2^2 \\ \text{s.t.} \quad & \dot{h}(x,u) \geq -\alpha\big(h(x)\big). \end{aligned} \tag{12}$$

Furthermore, the explicit solution of (12) *can be found by the Karush–Kuhn–Tucker (KKT)[60] conditions as[38]:*

$$k(x) = \begin{cases} k_{\rm des}(x) & \text{if } \varphi(x) \geq 0, \\ k_{\rm des}(x) - \frac{\varphi(x)\varphi_0^\top(x)}{\varphi_0(x)\varphi_0^\top(x)} & \text{otherwise,} \end{cases} \tag{13}$$

[2] In the literature, the term control barrier function is often used for functions that go to infinity at the boundary of the safe set, whereas functions that are zero at the boundary shall be called control safety functions. However, these terminologies are often used interchangeably in the literature, hence we use the more popular term, CBF.

where $\varphi(x) = L_f h(x) + L_g h(x)\, k_{\rm des}(x) + \alpha\big(h(x)\big)$ *and* $\varphi_0(x) = L_g h(x)$.

We remark that, with some extra care, the CBF condition (9) could be prescribed for $\forall x \in \mathbb{R}^n$ (rather than for $\forall x \in S$ only). This results in the attractivity of S, i.e., CBFs render S stabilizable, allowing the system to return to safe states from unsafe ones[61]. Furthermore, notice that inequality (9) is equivalent to:

$$L_g h(x) = 0 \;\Rightarrow\; L_f h(x) > -\alpha\big(h(x)\big). \tag{14}$$

When $L_g h(x) = 0$, the derivative of h with respect to time is not affected by the control input u according to (10), hence the corresponding uncontrolled system must be safe on its own. One may sufficiently satisfy (14) by requiring that $L_g h(x)$ is never zero. Then, the first derivative of h is always affected by the control input u, which is referred to as h has *relative degree* 1. Otherwise, higher derivatives of h may be affected by u that leads to higher relative degrees, given by the following definition.

Definition 4 (**Relative Degree**[62]**).** Function $h:\, \mathbb{R}^n \to \mathbb{R}$ has **relative degree** r (where $r \in \mathbb{Z}$, $r \geq 1$) w.r.t. (7) if it is r times continuously differentiable and the following holds $\forall x \in \mathbb{R}^n$:

$$\begin{aligned} L_g L_f^{r-1} h(x) &\neq 0, \\ L_g L_f^{k} h(x) &= 0, \quad \text{for } r \geq 2,\ k \in \{0, \dots, r-2\}, \end{aligned} \tag{15}$$

where $L_g L_f^0 h(x) = L_g h(x)$ and the second condition only applies for $r \geq 2$.

If h has a higher relative degree, it may not be a valid CBF. However, there exist systematic methods to construct CBFs from h and its derivatives, and guarantee safety; see[63,64,65,66] for details. For example, a popular method for the case of relative degree 2, where $L_g h(x) = 0$ and $L_g L_f h(x) \neq 0$, is the introduction of the *extended control barrier function*:

$$h_{\rm e}(x) = \underbrace{\dot{h}(x)}_{L_f h(x)} + \alpha\big(h(x)\big), \tag{16}$$

that has relative degree 1 and is a valid CBF. The process of extending CBFs for higher relative degrees ($r > 2$) is similar but more notationally intensive.

This highlights that the relative degree is an important property in safety-critical control. We will show that the relative degree may be significantly affected when time delay is included in the system[67]. Furthermore, we will accommodate the above extension method to time delay systems in Section 4.1 and apply it in examples in Section 5.1 (cases 3 and 4) and Section 5.2.

3 | SAFETY OF TIME DELAY SYSTEMS

In what follows, we extend the above concepts to certify safety and provide safety-critical controllers for time delay systems. We focus on so-called retarded and neutral type systems. If the rate of change of state depends on the past states of the system, then the corresponding mathematical model is a retarded functional differential equation (RFDE). If the rate of change of state depends on its own past values as well, then the corresponding equation is a neutral functional differential equation (NFDE).

To discuss safety for RFDEs and NFDEs, we denote the delay as $\tau > 0$, and we will rely on the following function spaces[68].

- $\mathcal{C}$: the space of continuous functions mapping from $[-\tau, 0]$ to $\mathbb{R}^n$ with norm $\|\phi\| = \max_{\vartheta\in[-\tau,0]} \|\phi(\vartheta)\|_2$ for $\phi \in \mathcal{C}$.
- $\mathcal{Q}$: the space of functions mapping from $[-\tau, 0]$ to $\mathbb{R}^n$ that are continuous almost everywhere on $[-\tau, 0]$ except possibly for a finite set of points with discontinuity of the first kind. The corresponding norm is $\|\phi\| = \sup_{\vartheta\in[-\tau,0]} \|\phi(\vartheta)\|_2$ for $\phi \in \mathcal{Q}$.
- $\mathcal{B}$: the space of continuous functions with almost everywhere continuous derivatives, i.e., $\mathcal{B} = \{\phi \in \mathcal{C} \,:\, \dot{\phi} \in \mathcal{Q}\}$, equipped with the norm $\|\phi\| = \max_{\vartheta\in[-\tau,0]} \|\phi(\vartheta)\|_2$ for $\phi \in \mathcal{B}$. Throughout the paper we use the convention that the derivative $\dot{\phi}$ indicates right-hand derivative at the discontinuity points.

3.1 | Retarded Dynamical Systems

First, we consider dynamical systems with time delays where the rate of change of the state depends on the past values of the state. This class of systems is referred as retarded functional differential equation and is given in the form:

$$\dot{x}(t) = \mathcal{F}(x_t), \tag{17}$$

where $x \in \mathbb{R}^n$ is the state variable and $x_t : [-\tau, 0] \to \mathbb{R}^n$ represents the history of the state over $[t-\tau, t]$ with delay $\tau > 0$. According to the standard Hale-Krasovsky notation[43,44], it is defined by the shift:

$$x_t(\vartheta) = x(t+\vartheta), \ \vartheta \in [-\tau, 0], \tag{18}$$

that is an element of the Banach space $\mathcal{B}$ defined above. Functional $\mathcal{F} : \mathcal{B} \to \mathbb{R}^n$ is assumed to be locally Lipschitz continuous, hence (17) has a unique solution over a time interval $t \in I(x_0)$ for any initial state history $x_0 \in \mathcal{B}$[69]. Again, for simplicity of exposition, we assume $I(x_0) = \mathbb{R}_{\geq 0}$. We define the derivative of the solution by $\dot{x}_t \in \mathcal{Q}$:

$$\dot{x}_t(\vartheta) = \begin{cases} \mathcal{F}(x_t) & \text{if } \vartheta = 0, \\ \dot{x}(t+\vartheta) & \text{if } \vartheta \in [-\tau, 0). \end{cases} \tag{19}$$

Since the state space of (17) is $\mathcal{B}$, which is infinite dimensional, one needs the state x_t to evolve within an infinite dimensional *safe set* $\mathcal{S} \subset \mathcal{B}$ to certify safety as given by the following definition.

Definition 5 (Safety and Forward Invariance of RFDE). System (17) is safe w.r.t. set $\mathcal{S} \subset \mathcal{B}$, if $\mathcal{S}$ is forward invariant w.r.t. (17) such that $x_0 \in \mathcal{S} \implies x_t \in \mathcal{S}, \forall t \geq 0$ for the solution of (17).

Assume that the set $\mathcal{S}$ can be constructed as the 0-superlevel set of a continuously Fréchet differentiable functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ (rather than a function):

$$\mathcal{S} = \{x_t \in \mathcal{B} : \mathcal{H}(x_t) \geq 0\}. \tag{20}$$

The forward invariance of set $\mathcal{S}$ implies that if the initial condition x_0 is within the safe set and satisfies $\mathcal{H}(x_0) \geq 0$, then the solution x_t stays in the safe set for all time and satisfies $\mathcal{H}(x_t) \geq 0$, $\forall t \geq 0$. This makes it evident that safety functions defined over the finite dimensional space $\mathbb{R}^n$ are not adequate to establish safety. Instead, one may construct so-called *safety functionals* defined over the infinite dimensional state space $\mathcal{B}$.

Definition 6 (Safety Functional[51]). A continuously Fréchet differentiable functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ is a **safety functional** for (17) on $\mathcal{S}$ defined by (20) if there exists $\alpha \in \mathcal{K}_\infty^{\mathrm{e}}$ such that $\forall x_t \in \mathcal{B}$:

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) \geq -\alpha\big(\mathcal{H}(x_t)\big), \tag{21}$$

where $\dot{\mathcal{H}} := \mathcal{L}_{\mathcal{F}}\mathcal{H} : \mathcal{B} \times \mathcal{Q} \to \mathbb{R}$ is the derivative of $\mathcal{H}$ along (17) with $\dot{x}_t$ given in (19).

Note that the derivative of $\mathcal{H}$ depends both on x_t and $\dot{x}_t$ in (19), and its calculation is detailed below. The following theorem from[51] certifies safety, i.e., the forward invariance of $\mathcal{S}$ via safety functionals.

Theorem 3. [51] *Set $\mathcal{S}$ in* (20) *is forward invariant w.r.t.* (17) *if $\mathcal{H}$ is a safety functional for* (17) *on $\mathcal{S}$, i.e.,* (21) *is satisfied.*

Proof[51]. The proof follows from the comparison lemma. We set up the initial value problem (with $y \in \mathbb{R}$):

$$\dot{y}(t) = -\alpha\big(y(t)\big), \qquad y(0) = \mathcal{H}(x_0), \tag{22}$$

which has the following unique solution (see the proof of Theorem 2):

$$y(t) = \beta\big(\mathcal{H}(x_0), t\big) \tag{23}$$

for $t \geq 0$, where $\beta \in \mathcal{K}_\infty^{\mathrm{e}}\mathcal{L}$. Then, applying the comparison lemma for (21) and (22), we obtain:

$$\mathcal{H}(x_t) \geq \beta\big(\mathcal{H}(x_0), t\big), \quad \forall t \geq 0. \tag{24}$$

This leads to $\mathcal{H}(x_0) \geq 0 \implies \mathcal{H}(x_t) \geq 0$, $\forall t \geq 0$, that is, $\mathcal{S}$ is forward invariant w.r.t. (17). This completes the proof. □

3.2 | Time Derivative of Safety Functional

The left-hand side of (21) is the time derivative of $\mathcal{H}$ along the solution of (17). While for finite dimensional delay-free systems the derivative $\dot{h}$ is given by the directional derivative or Lie derivative[43,70], the derivative $\dot{\mathcal{H}}$ in the presence of time delay and infinite dimensional dynamics has an intricate representation which we break down below.

Recall that in the delay-free case, the derivative $\dot{h}$ of the safety function is calculated by a scalar product as $\dot{h}(x) = \nabla h(x)\dot{x}$. That is, $\dot{h}$ is given by a linear function of $\dot{x}$ where $\dot{x} = f(x)$. For time delay systems, the derivative $\dot{\mathcal{H}}$ of the safety functional is calculated by an integral and it is given by a linear functional of $\dot{x}_t$ in (19). This is stated by the theorem below.

Theorem 4. *Consider system* (17) *and let* $\mathcal{H}: \mathcal{B} \to \mathbb{R}$ *be a continuously Fréchet differentiable functional. Then there exists a unique* $\eta: \mathcal{B} \times [-\tau, 0] \to \mathbb{R}^{1\times n}$ *that is of bounded variation*[69,47] *in its second argument such that the time derivative of* $\mathcal{H}$ *along* (17) *can be expressed as:*

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) = \int_{-\tau}^{0} \mathrm{d}_\vartheta \eta(x_t, \vartheta)\dot{x}_t(\vartheta), \tag{25}$$

where the integral is a Stieltjes type.

Proof. Here we provide the main steps of the proof and the remaining details (definitions and a lemma) are in Appendix B. The derivative of $\mathcal{H}$ along (17) can be expressed by the Gâteaux derivative (see definition at (B4)) along $\dot{x}_t$ as:

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) = \lim_{\Delta t \to 0} \frac{\mathcal{H}(x_{t+\Delta t}) - \mathcal{H}(x_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathcal{H}(x_t + \Delta t \dot{x}_t) - \mathcal{H}(x_t)}{\Delta t} = D_{\mathrm{G}}\mathcal{H}(x_t)(\dot{x}_t), \tag{26}$$

where $D_{\mathrm{G}}\mathcal{H}(x_t): \mathcal{Q} \to \mathbb{R}$ denotes the Gâteaux derivative at x_t. Since $\mathcal{H}$ is continuously Fréchet differentiable, the Gâteaux derivative exists and can be given by the Fréchet derivative (see definition at (B5)) based on Lemma 1:

$$D_{\mathrm{G}}\mathcal{H}(x_t)(\dot{x}_t) = D_{\mathrm{F}}\mathcal{H}(x_t)\dot{x}_t. \tag{27}$$

Here $D_{\mathrm{F}}\mathcal{H}(x_t): \mathcal{Q} \to \mathbb{R}$ is the Fréchet derivative at x_t that is a continuous linear functional (applied on $\dot{x}_t$). Then, the Riesz representation theorem (see Theorem 7 in Appendix B) implies that there exists a unique $\eta: \mathcal{B} \times [-\tau, 0] \to \mathbb{R}^{1\times n}$ that is of bounded variation[69,47] in its second argument such that this linear functional can be expressed as the Stieltjes integral:

$$D_{\mathrm{F}}\mathcal{H}(x_t)\dot{x}_t = \int_{-\tau}^{0} \mathrm{d}_\vartheta \eta(x_t, \vartheta)\dot{x}_t(\vartheta), \tag{28}$$

which leads to (25) and proves the statement in Theorem 4. □

As such, the integral in (25) is the infinite dimensional counterpart of the scalar product $\nabla h(x)\dot{x}$, while $\mathrm{d}_\vartheta \eta(x_t, .)$ represents the infinite dimensional counterpart of the gradient $\nabla h(x)$. In the delay-free case, the function h is assumed to be continuously differentiable, hence the gradient $\nabla h(x)$ exists, and it allows the calculation of the directional derivatives of h in any direction, including $\dot{x}$. With delay, we assume that the functional $\mathcal{H}$ is continuously Fréchet differentiable, and hence the Gâteaux derivative can be expressed along any direction, including $\dot{x}_t$. Furthermore, the integral in (25) can also be expressed as a distribution:

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) = \int_{-\infty}^{\infty} w(x_t, \vartheta)\tilde{\dot{x}}_t(\vartheta)\,\mathrm{d}\vartheta, \tag{29}$$

with a kernel $w: \mathcal{B} \times \mathbb{R} \to \mathbb{R}^{1\times n}$ and $\tilde{\dot{x}}_t: \mathbb{R} \to \mathbb{R}^n$ defined by $\tilde{\dot{x}}_t(\vartheta) = \dot{x}_t(\vartheta)$ if $\vartheta \in [-\tau, 0]$ and $\tilde{\dot{x}}_t(\vartheta) = 0$ otherwise.

The expressions of η in (25) and w in (29) depend on the specific form of $\mathcal{H}$ (just as the expression of ∇h depends on the form of h). We demonstrate the calculation of these expressions and the time derivative of the functional $\mathcal{H}$ by an example below, which covers most of the scenarios that appear in practical applications.

Example 1. Consider the system (17) with functional $\mathcal{H}$ that contains both delay-free states, discrete (point) delays $\tau_j \in [-\tau, 0]$, $j \in \{1, \dots, l\}$, and a continuous (distributed) delay over $[-\sigma_1, -\sigma_2] \subseteq [-\tau, 0]$, defined as:

$$\mathcal{H}(x_t) = h\left(x_t(0), x_t(-\tau_1), \dots, x_t(-\tau_l), \int_{-\sigma_1}^{-\sigma_2} \rho(\vartheta)\,\kappa\big(x_t(\vartheta)\big)\,\mathrm{d}\vartheta \right), \tag{30}$$

where $h: \mathbb{R}^n \times \dots \times \mathbb{R}^n \to \mathbb{R}$, $\kappa: \mathbb{R}^n \to \mathbb{R}^n$ and $\rho: [-\sigma_1, -\sigma_2] \to \mathbb{R}^{n\times n}$ are continuously differentiable. One may directly take the time derivative of (30) and substitute (19), which leads to the following derivative required for certifying safety:

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) = \underbrace{\nabla_0 h(\dots)}_{w_0(x_t)} \mathcal{F}(x_t) + \sum_{j=1}^{l} \underbrace{\nabla_j h(\dots)}_{w_j(x_t)} \dot{x}_t(-\tau_j) + \int_{-\sigma_1}^{-\sigma_2} \underbrace{\nabla_{l+1} h(\dots)\rho(\vartheta)\nabla\kappa\big(x_t(\vartheta)\big)}_{w_{\mathrm{d}}(x_t,\vartheta)} \dot{x}_t(\vartheta)\,\mathrm{d}\vartheta, \tag{31}$$

where ∇_j is the gradient with respect to the j-th argument of a function, $\nabla\kappa$ is the Jacobian of κ, and $(\dots)$ is a shorthand notation for evaluation at the argument of h as in (30). The derivative $\dot{\mathcal{H}}$ can be written in the form (29), where kernel w comprises of

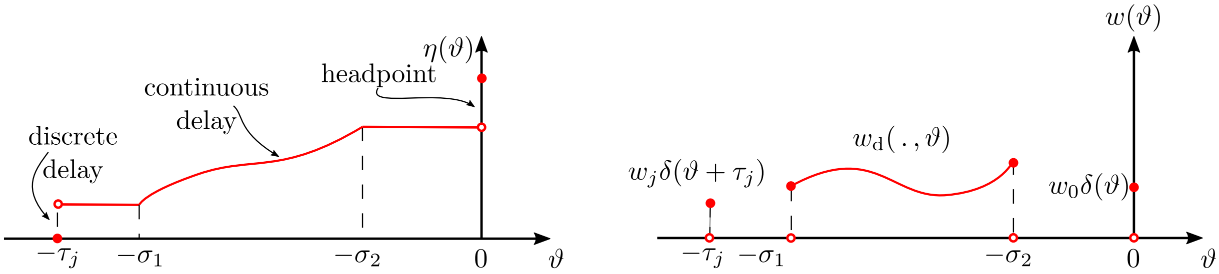

FIGURE 2 Illustration of the bounded variation function η in (34) and the distribution kernel w in (32) for a discrete delay τ_j and a continuous delay over $[-\sigma_1, -\sigma_2]$.

finitely many shifted Dirac delta distributions (corresponding to the point delays) and a bounded kernel (corresponding to the remaining distributed delay):

$$w(x_t, \vartheta) = w_0(x_t)\delta(\vartheta) + \sum_{j=1}^{l} w_j(x_t)\delta(\vartheta + \tau_j) + \tilde{w}_{\rm d}(x_t, \vartheta), \tag{32}$$

with δ being the Dirac delta and:

$$\tilde{w}_{\rm d}(x_t, \vartheta) = \begin{cases} 0 & \text{if} \quad \vartheta < -\sigma_1, \\ w_{\rm d}(x_t, s) & \text{if} \quad -\sigma_1 \le \vartheta \le -\sigma_2, \\ 0 & \text{if} \quad -\sigma_2 < \vartheta. \end{cases} \tag{33}$$

The corresponding bounded variation in (25) has the form:

$$\eta(x_t, \vartheta) = w_0(x_t)\theta(\vartheta) + \sum_{j=1}^{l} w_j(x_t)\hat{\theta}(\vartheta + \tau_j) + \eta_{\rm d}(x_t, \vartheta). \tag{34}$$

Here θ and $\hat{\theta}$ denote the right and left continuous Heaviside step functions:

$$\theta(s) = \begin{cases} 1 & \text{if} \quad s \ge 0, \\ 0 & \text{if} \quad s < 0, \end{cases} \qquad \hat{\theta}(s) = 1 - \theta(-s) = \begin{cases} 1 & \text{if} \quad s > 0, \\ 0 & \text{if} \quad s \le 0, \end{cases} \tag{35}$$

and:

$$\eta_{\rm d}(x_t, \vartheta) = \begin{cases} 0 & \text{if} \quad \vartheta < -\sigma_1, \\ \int_{-\sigma_1}^{\vartheta} w_{\rm d}(x_t, s)\,{\rm d}s & \text{if} \quad -\sigma_1 \le \vartheta \le -\sigma_2, \\ \int_{-\sigma_1}^{-\sigma_2} w_{\rm d}(x_t, s)\,{\rm d}s & \text{if} \quad -\sigma_2 < \vartheta. \end{cases} \tag{36}$$

Function η in (34) and the kernel w in (32) are illustrated in Fig. 2. We summarize the conclusions about Example 1 as follows.

Corollary 2. *When $\mathcal{H}(x_t)$ involves both delay-free states, point delays $\tau_j \in [-\tau, 0]$, $j \in \{1, \dots, l\}$, and a distributed delay over $[-\sigma_1, -\sigma_2] \subseteq [-\tau, 0]$, the time derivative of $\mathcal{H}$ in* (25) *along* (19) *is of the form:*

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) = \mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t, \dot{x}_t) := w_0(x_t)\mathcal{F}(x_t) + \sum_{j=1}^{l} w_j(x_t)\dot{x}_t(-\tau_j) + \int_{-\sigma_1}^{-\sigma_2} w_{\rm d}(x_t, \vartheta)\dot{x}_t(\vartheta)\,{\rm d}\vartheta, \tag{37}$$

with weights $w_0, w_j : \mathcal{B} \to \mathbb{R}^{1\times n}$ and $w_{\rm d} : \mathcal{B} \times \mathbb{R} \to \mathbb{R}^{1\times n}$ that depend on the specific expression of $\mathcal{H}$.

This leads to two important properties about the derivative of $\mathcal{H}$, given by the following remarks. We will rely on these properties when discussing the relative degree of control barrier functionals in Section 4.1.

Remark 1. ($\mathcal{H}(x_t)$ contains $x(t)$). When $\mathcal{H}(x_t)$ contains the present state $x(t) = x_t(0)$, it is indicated by $w_0(x_t) \neq 0$, $\forall x_t \in \mathcal{B}$. Then, the time derivative $\dot{\mathcal{H}}(x_t, \dot{x}_t)$ is directly affected by the right-hand side $\mathcal{F}(x_t)$. This will be a necessary requirement for enforcing safety via control (i.e., when we maintain safety by designing $\mathcal{F}$ for a closed control loop).

Remark 2. ($\mathcal{L}_{\mathcal{F}}\mathcal{H}$ independent of $\dot{x}_t$). If $w_{\rm d}$ is continuously differentiable in ϑ, with derivative denoted by $w_{\rm d}'$, then the integral $\int_{-\sigma_1}^{-\sigma_2} w_{\rm d}(x_t, \vartheta)\dot{x}_t(\vartheta){\rm d}\vartheta$ can be simplified. Since $\frac{\partial}{\partial t}x_t(\vartheta) = \frac{\partial}{\partial \vartheta}x_t(\vartheta)$, integration by parts eliminates $\dot{x}_t$ and leads to an expression that depends on x_t only:

$$\Lambda(x_t) := \int_{-\sigma_1}^{-\sigma_2} w_{\mathrm{d}}(x_t,\vartheta)\dot{x}_t(\vartheta)\,\mathrm{d}\vartheta = w_{\mathrm{d}}(x_t,-\sigma_2)x_t(-\sigma_2) - w_{\mathrm{d}}(x_t,-\sigma_1)x_t(-\sigma_1) - \int_{-\sigma_1}^{-\sigma_2} w'_{\mathrm{d}}(x_t,\vartheta)x_t(\vartheta)\,\mathrm{d}\vartheta. \tag{38}$$

Additionally, if $w_j(x_t) = 0$, $\forall x_t \in \mathcal{B}$, $\forall j \geq 1$ also holds, then $\mathcal{L}_{\mathcal{F}}\mathcal{H}$ in (37) can be expressed as a functional of x_t only, independent of $\dot{x}_t$, which we shorty denote as $\mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t)$. Then, the time derivative of $\mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t)$ can be calculated by the same method as that of $\mathcal{H}(x_t)$. We will exploit this in higher relative degree scenarios in Section 4.1.

While Example 1 covers most practical choices of safety functionals, it does not include all possibilities for $\mathcal{H}$. A more general example with a double integral can be found in Appendix C, and one could also include triple, quadruple, etc. integrals.

3.3 | Neutral Dynamical Systems

Now, we address time delay systems where the rate of change of the state depends on the past values of the state as well as past state derivatives. This class of systems is referred as neutral functional differential equation (NFDE) and is given in the form:

$$\dot{x}(t) = \mathcal{F}(x_t, \dot{x}_t), \tag{39}$$

where $x \in \mathbb{R}^n$ is the state variable, $x_t \in \mathcal{B}$ is the state history defined in (18) and $\dot{x}_t \in \mathcal{Q}$ is its derivative. The functional $\mathcal{F} : \mathcal{B} \times \mathcal{Q} \to \mathbb{R}^n$ is assumed to be locally Lipschitz continuous in its first argument (x_t), continuous in its second argument ($\dot{x}_t$), and assumed to satisfy a modified Lipschitz condition that takes the form:

$$\|\mathcal{F}\big(x_t,\phi_1\big) - \mathcal{F}\big(x_t,\phi_2\big)\|_2 \leq L\|\phi_1 - \phi_2\|, \quad L < 1, \tag{40}$$

for any functions $\phi_1, \phi_2 \in \mathcal{Q}$ such that $\phi_1(\vartheta) = \phi_2(\vartheta)$, $\forall \vartheta \in [-\tau, -\delta]$ for some $\delta > 0$.

According to Theorem 3.2 in[71], if the condition (40) holds, system (39) has a unique (continuous but potentially nonsmooth) solution over a time interval $t \in I(x_0, \dot{x}_0)$ for initial history $x_0 \in \mathcal{B}$ and $\dot{x}_0 \in \mathcal{Q}$. This existence and uniqueness result was first proved by[72] and later also summarized in[73,74]. The derivative of the solution then becomes:

$$\dot{x}_t(\vartheta) = \begin{cases} \mathcal{F}(x_t,\dot{x}_t) & \text{if } \vartheta = 0, \\ \dot{x}(t+\vartheta) & \text{if } \vartheta \in [-\tau, 0). \end{cases} \tag{41}$$

For comprehensive details on the properties of NFDEs, please refer to[43,44].

The safety of (39) can be formulated the same way as it was done for retarded systems in Section 3.1. Safety means that the state x_t evolves within the safe set $\mathcal{S} \subset \mathcal{B}$, which is constructed by the safety functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ (see Definitions 5 and 6). Then, the safety of the neutral system (39) is formally certified by the following corollary.

Corollary 3. *Set $\mathcal{S}$ in* (20) *is forward invariant w.r.t.* (39) *if $\mathcal{H}$ is a safety functional for* (39) *on $\mathcal{S}$, i.e.:*

$$\dot{\mathcal{H}}(x_t, \dot{x}_t) \geq -\alpha\big(\mathcal{H}(x_t)\big), \tag{42}$$

is satisfied, where $\dot{\mathcal{H}} := \mathcal{L}_{\mathcal{F}}\mathcal{H} : \mathcal{B} \times \mathcal{Q} \to \mathbb{R}$ is the derivative of $\mathcal{H}$ along (39) *with $\dot{x}_t$ given in* (41).

The difference between retarded and neutral systems is that for the neutral case the derivative of $\mathcal{H}$ is provided by substituting (41) into (25) (while we used (19) for retarded systems). Analogously to (37), we can write:

$$\dot{\mathcal{H}}(x_t,\dot{x}_t) = w_0(x_t)\mathcal{F}(x_t,\dot{x}_t) + \sum_{j=1}^{l} w_j(x_t)\dot{x}_t(-\tau_j) + \int_{-\sigma_1}^{-\sigma_2} w_{\mathrm{d}}(x_t,\vartheta)\dot{x}_t(\vartheta)\,\mathrm{d}\vartheta. \tag{43}$$

4 | SAFETY-CRITICAL CONTROL OF TIME DELAY SYSTEMS

Building upon the framework establishing safety for dynamical systems with time delays, we now extend this approach to control systems with time delays. Similarly to how safety functions were extended to control barrier functions in Section 2, in this section we extend safety functionals to control barrier functionals, and use them as tool for safety-critical controller synthesis.

Let us consider the following affine control system with state delay:

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)u(t), \tag{44}$$

where $x \in \mathbb{R}^n$ is the state, $x_t \in \mathcal{B}$ is the state history defined in (18), $\dot{x}_t \in \mathcal{Q}$ is its derivative, $u \in \mathbb{R}^m$ is the input, while $\mathcal{F} : \mathcal{B} \to \mathbb{R}^n$ and $\mathcal{G} : \mathcal{B} \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous functionals in their first argument and continuous in their second. The state derivative along this system can also be expressed as:

$$\dot{x}_t(\vartheta) = \begin{cases} \mathcal{F}(x_t) + \mathcal{G}(x_t)u(t), & \text{if } \vartheta = 0, \\ \dot{x}(t+\vartheta) & \text{if } \vartheta \in [-\tau, 0). \end{cases} \tag{45}$$

We seek to design a locally Lipschitz continuous controller $\mathcal{K} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^m$, $u = \mathcal{K}(x_t, \dot{x}_t)$, and enforce the closed-loop system:

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)\mathcal{K}(x_t, \dot{x}_t) \tag{46}$$

to be safe w.r.t. the set $\mathcal{S}$ in (20), that is, $\mathcal{S}$ is forward invariant w.r.t. (46). The overall right-hand side of the closed-loop system (46) should satisfy the same restrictions as $\mathcal{F}$ in (40) to have existence and uniqueness guarantee.

To design a control input that guarantees the system to be safe motivates the introduction of *control barrier functionals*.

Definition 7 (**Control Barrier Functional, CBFal**). A continuously Fréchet differentiable functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ is a **control barrier functional** (CBFal) for (44) on $\mathcal{S}$ defined by (20), if there exists $\alpha \in \mathcal{K}^{\mathrm{e}}_\infty$ such that $\forall x_t \in \mathcal{B}$:

$$\sup_{u\in\mathbb{R}^m} \dot{\mathcal{H}}(x_t, \dot{x}_t, u) > -\alpha\big(\mathcal{H}(x_t)\big), \tag{47}$$

where

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, u) = \mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t, \dot{x}_t) + \mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t)\, u \tag{48}$$

is the derivative of $\mathcal{H}$ along (44), given by $\dot{x}_t$ in (45) and the functionals $\mathcal{L}_{\mathcal{F}}\mathcal{H} : \mathcal{B} \to \mathbb{R}$ and $\mathcal{L}_{\mathcal{G}}\mathcal{H} : \mathcal{B} \to \mathbb{R}^{1\times m}$.

Note that the derivative $\dot{\mathcal{H}}$ is still obtained by the linear functional in (25), however, $\dot{x}_t$ is now given by (45) and it involves $u(t)$. Therefore, $\dot{\mathcal{H}}$ in (47) depends on u as an affine function. Analogously to (37), the time derivative of $\mathcal{H}$ is expressed as follows for the case of multiple point delays and an additional distributed delay:

$$\begin{aligned} \mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t, \dot{x}_t) =& w_0(x_t)\mathcal{F}(x_t) + \sum_{j=1}^{l} w_j(x_t)\dot{x}_t(-\tau_j) + \int_{-\sigma_1}^{-\sigma_2} w_{\mathrm{d}}(x_t, \vartheta)\dot{x}_t(\vartheta)\,\mathrm{d}\vartheta, \\ \mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t) =& w_0(x_t)\mathcal{G}(x_t). \end{aligned} \tag{49}$$

With the definition of CBFal we state our main result to ensure safety for systems with state delay by extending Theorem 3.

Theorem 5. *If $\mathcal{H}$ is a CBFal for* (44) *on $\mathcal{S}$ defined by* (20)*, then any locally Lipschitz continuous controller $\mathcal{K} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^m$, $u = \mathcal{K}(x_t, \dot{x}_t)$ satisfying:*

$$\dot{\mathcal{H}}\big(x_t, \dot{x}_t, \mathcal{K}(x_t, \dot{x}_t)\big) \geq -\alpha\big(\mathcal{H}(x_t)\big). \tag{50}$$

$\forall x_t \in \mathcal{S}$ *renders set $\mathcal{S}$ forward invariant w.r.t.* (46).

Proof. Definition 7 ensures that controller $\mathcal{K}$ exists. Then considering the closed-loop dynamics in (46), the set $\mathcal{S}$ is forward invariant according to Theorem 3. □

This result motivates the construction of pointwise optimal safety-critical controllers that use the nearest safe action to a nominal but potentially unsafe controller.

Corollary 4. *Given a CBFal $\mathcal{H}$ and a locally Lipschitz continuous desired controller $\mathcal{K}_{\mathrm{des}} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^m$, $u_{\mathrm{des}} = \mathcal{K}_{\mathrm{des}}(x_t, \dot{x}_t)$ under the condition* (40)*, the following quadratic program (QP) yields a controller $\mathcal{K} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^m$, $u = \mathcal{K}(x_t, \dot{x}_t)$ that renders set $\mathcal{S}$ in* (20) *forward invariant w.r.t.* (46)*:*

$$\begin{aligned} \mathcal{K}(x_t, \dot{x}_t) = \underset{u\in\mathbb{R}^m}{\operatorname{argmin}} \quad & \frac{1}{2}\|u - \mathcal{K}_{\mathrm{des}}(x_t, \dot{x}_t)\|_2^2 \\ \text{s.t.} \quad & \dot{\mathcal{H}}(x_t, \dot{x}_t, u) \geq -\alpha\big(\mathcal{H}(x_t)\big). \end{aligned} \tag{51}$$

Furthermore, the explicit solution of (51) *can be found by the Karush–Kuhn–Tucker (KKT)*[60] *conditions as:*

$$\mathcal{K}(x_t,\dot{x}_t) = \begin{cases} \mathcal{K}_{\mathrm{des}}(x_t,\dot{x}_t) & \text{if } \phi(x_t,\dot{x}_t) \geq 0, \\ \mathcal{K}_{\mathrm{des}}(x_t,\dot{x}_t) - \frac{\phi(x_t,\dot{x}_t)\phi_0^\top(x_t)}{\phi_0(x_t)\phi_0^\top(x_t)} & \text{otherwise,} \end{cases} \tag{52}$$

where $\phi(x_t,\dot{x}_t) = \mathcal{L}_\mathcal{F}\mathcal{H}(x_t,\dot{x}_t) + \mathcal{L}_\mathcal{G}\mathcal{H}(x_t)\,\mathcal{K}_{\mathrm{des}}(x_t,\dot{x}_t) + \alpha\left(\mathcal{H}(x_t)\right)$ *and* $\phi_0(x_t) = \mathcal{L}_\mathcal{G}\mathcal{H}(x_t)$*.*

The derivation of (52) is detailed in Appendix D. This allows operating the nominal controller when it is safe ($\phi \geq 0$), and modifies the input to keep the system safe otherwise ($\phi < 0$).

Remark 3. (Control of neutral systems). Note that while the control system (44) contains retarded terms (functionals of x_t) only, the closed-loop system (46) is neutral (includes $\dot{x}_t$). Therefore, one may extend the theory to neutral control systems of the form:

$$\dot{x}(t) = \mathcal{F}(x_t,\dot{x}_t) + \mathcal{G}(x_t,\dot{x}_t)u(t), \tag{53}$$

where $\mathcal{F} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^n$ and $\mathcal{G} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous functionals in their first argument and continuous in their second. The corresponding closed-loop system:

$$\dot{x}(t) = \mathcal{F}(x_t,\dot{x}_t) + \mathcal{G}(x_t,\dot{x}_t)\mathcal{K}(x_t,\dot{x}_t) \tag{54}$$

is still neutral, and its right-hand side should satisfy the same restrictions as $\mathcal{F}$ in (40) to have existence and uniqueness guarantee. For control design, we further require that $\mathcal{F}(x_t,\dot{x}_t)$ and $\mathcal{G}(x_t,\dot{x}_t)$ do not include $\dot{x}_t(0) = \dot{x}(t)$, i.e., that (53) expresses $\dot{x}(t)$ explicitly (not implicitly). Then, safety-critical controllers can still be synthesized using (50), where the derivative of $\mathcal{H}$ is:

$$\dot{\mathcal{H}}(x_t,\dot{x}_t,u) = \mathcal{L}_\mathcal{F}\mathcal{H}(x_t,\dot{x}_t) + \mathcal{L}_\mathcal{G}\mathcal{H}(x_t,\dot{x}_t)\,u \tag{55}$$

cf. (48), with $\mathcal{L}_\mathcal{F}\mathcal{H} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}$ and $\mathcal{L}_\mathcal{G}\mathcal{H} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^{1\times m}$ given as follows in the case of multiple point delays and an additional distributed delay:

$$\begin{aligned} \mathcal{L}_\mathcal{F}\mathcal{H}(x_t,\dot{x}_t) =& w_0(x_t)\mathcal{F}(x_t,\dot{x}_t) + \sum_{j=1}^{l} w_j(x_t)\dot{x}_t(-\tau_j) + \int_{-\sigma_1}^{-\sigma_2} w_\mathrm{d}(x_t,\vartheta)\dot{x}_t(\vartheta)\,\mathrm{d}\vartheta, \\ \mathcal{L}_\mathcal{G}\mathcal{H}(x_t,\dot{x}_t) =& w_0(x_t)\mathcal{G}(x_t,\dot{x}_t), \end{aligned} \tag{56}$$

cf. (49). Note that in this case $\mathcal{L}_\mathcal{F}\mathcal{H}$ may not be independent of $\dot{x}_t$ even if the conditions in Remark 2 hold due to the occurrence of $\dot{x}_t$ in $\mathcal{F}$. Furthermore, $\mathcal{L}_\mathcal{G}\mathcal{H}$ also depends on $\dot{x}_t$ through $\mathcal{G}$.

Corollary 5. *If* $\mathcal{H}$ *is a CBFal for* (53) *on* $\mathcal{S}$ *defined by* (20)*, then any locally Lipschitz continuous controller* $\mathcal{K} : \mathcal{B}\times\mathcal{Q} \to \mathbb{R}^m$*,* $u = \mathcal{K}(x_t,\dot{x}_t)$ *satisfying* (50) $\forall x_t \in \mathcal{S}$ *renders set* $\mathcal{S}$ *forward invariant w.r.t.* (54).

In the presence of time delay, the relative degree is affected by the delay itself, which is detailed next. For simplicity, we omit further discussions on neutral control systems, and the rest of the paper addresses the retarded control system (44).

4.1 | Delay-induced Higher Relative Degree

The CBFal condition (47) sufficiently holds if $\mathcal{L}_\mathcal{G}\mathcal{H}(x_t) \neq 0$ for all $\forall x_t \in \mathcal{B}$. We refer to this as $\mathcal{H}$ having *relative degree* 1. Relative degree is an important concept for time delay systems, too, which motivates the extension of Definition 4.

Definition 8 (**Relative Degree of Functional**). Functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ has **relative degree** r (where $r \in \mathbb{Z}$, $r \geq 1$) w.r.t. (44) if it is r times continuously Fréchet differentiable and satisfies the two conditions below.

I. For $r \geq 2$, the Fréchet derivative of $\mathcal{L}_\mathcal{F}^k\mathcal{H}(x_t,\dot{x}_t)$ w.r.t. $\dot{x}_t$ is zero $\forall x_t \in \mathcal{B}$ and $\forall k \in \{1,\ldots,r-1\}$. That is, $\mathcal{L}_\mathcal{F}^k\mathcal{H}(x_t,\dot{x}_t)$ does not depend on $\dot{x}_t$, only on x_t, which we denote shortly as $\mathcal{L}_\mathcal{F}^k\mathcal{H}(x_t)$.

II. The following holds $\forall x_t \in \mathcal{B}$:

$$\begin{aligned} &\mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}^{r-1}\mathcal{H}(x_t) \neq 0, \\ &\mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}^{k}\mathcal{H}(x_t) = 0, \quad \text{for } r \geq 2,\ k \in \{0,\ldots,r-2\}, \end{aligned} \tag{57}$$

where $\mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}^0\mathcal{H}(x_t) = \mathcal{L}_\mathcal{G}\mathcal{H}(x_t)$ and the second condition only applies for $r \geq 2$.

Condition I is specific to time delay systems and does not have a delay-free counterpart in Definition 4. This condition is imposed because otherwise for higher relative degree, $r \geq 2$, higher time derivatives of $\mathcal{H}$ could include higher derivatives of x_t. Synthesizing a controller in this case could lead to a closed-loop system where the rate of change of state depends on past values of higher derivatives of the state (e.g., systems of the form $\dot{x}(t) = \mathcal{F}(x_t, \dot{x}_t, \ddot{x}_t)$), which is called advanced functional differential equation. We demonstrate this by an example in Section 5.1. Advanced type equations are rarely used in engineering applications due to their inverted causality problem[75], hence Condition I excludes this possibility.

Definition 8 with (49) leads to the following observations when a combination of point and distributed delays occurs in $\mathcal{H}$.

- For *relative degree* 1, $\mathcal{L}_\mathcal{G}\mathcal{H}(x_t) = w_0(x_t)\mathcal{G}(x_t) \neq 0$ that implies $w_0(x_t) \neq 0$. This means that $\mathcal{H}(x_t)$ contains $x(t)$ as discussed in Remark 1, i.e., the *CBFal includes the present state* $x(t)$.
- For *relative degree* 2, $\mathcal{L}_\mathcal{F}\mathcal{H}(x_t)$ is independent of $\dot{x}_t$. Per Remark 2, this implies $w_j(x_t) = 0$, $\forall x_t \in \mathcal{B}$, $\forall j \geq 1$, i.e., the *CBFal excludes states* $x(t-\tau_j)$ *with point delay*. Moreover, $\mathcal{L}_\mathcal{G}\mathcal{H}(x_t) = w_0(x_t)\mathcal{G}(x_t) = 0$ and $\mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}\mathcal{H}(x_t) \neq 0$. The former occurs if $w_0(x_t) = 0$, i.e., the CBFal does not contain the present state $x(t)$ only a distributed delay term. Alternatively, the expression of $\mathcal{G}$ may also cause $\mathcal{L}_\mathcal{G}\mathcal{H}(x_t) = 0$, even if $w_0(x_t) \neq 0$ and the CBFal includes the present state $x(t)$.
- There is *no valid relative degree* if $w_0(x_t) = 0$ and $w_j(x_t) \neq 0$ for some $j \geq 1$, i.e., when the *CBFal excludes the present state* $x(t)$ *but includes the past state* $x(t-\tau_j)$. That is, the past of a system cannot be rendered safe.

In conclusion, distributed delays in $\mathcal{H}$ may induce higher relative, whereas point delays may lead to no valid relative degree, if the present state does not explicitly appear in $\mathcal{H}$.

For higher relative degree problems in delay-free systems, controllers can be synthesized by constructing a relative degree 1 CBF from h and its derivatives; see the methods in[63,64,65,66] and the discussion at (16). This process can be extended to time delay systems, which we demonstrate for relative degree 2. We consider $\mathcal{L}_\mathcal{G}\mathcal{H}(x_t) = 0$ and that $\mathcal{L}_\mathcal{F}\mathcal{H}$ does not contain terms of $\dot{x}_t$. We introduce the *extended control barrier functional*:

$$\mathcal{H}_\mathrm{e}(x_t) = \underbrace{\dot{\mathcal{H}}(x_t)}_{\mathcal{L}_\mathcal{F}\mathcal{H}(x_t)} + \alpha\big(\mathcal{H}(x_t)\big), \tag{58}$$

that has relative degree 1 if $\mathcal{H}$ has relative degree 2 since $\mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}\mathcal{H}(x_t) \neq 0$. Its 0-superlevel set is denoted by:

$$\mathcal{S}_\mathrm{e} = \big\{x_t \in \mathcal{B} : \mathcal{H}_\mathrm{e}(x_t) \geq 0\big\}. \tag{59}$$

Definition 9 (**Extended Control Barrier Functional**). Let $\mathcal{H} : \mathcal{B} \to \mathbb{R}$ be a twice continuously Fréchet differentiable functional with continuously differentiable $\alpha \in \mathcal{K}_\infty^\mathrm{e}$, $\mathcal{L}_\mathcal{G}\mathcal{H}(x_t) = 0$ and continuously differentiable $\mathcal{L}_\mathcal{F}\mathcal{H}$ excluding terms of $\dot{x}_t$. Then functional $\mathcal{H}_\mathrm{e} : \mathcal{B} \to \mathbb{R}$ defined by (58) is an **extended control barrier functional** (extended CBFal) for (44) on $\mathcal{S} \cap \mathcal{S}_\mathrm{e}$ defined by (20) and (59), if there exists $\alpha_\mathrm{e} \in \mathcal{K}_\infty^\mathrm{e}$ such that $\forall x_t \in \mathcal{B}$:

$$\sup_{u \in \mathbb{R}^m} \dot{\mathcal{H}}_\mathrm{e}(x_t, \dot{x}_t, u) > -\alpha_\mathrm{e}\big(\mathcal{H}_\mathrm{e}(x_t)\big), \tag{60}$$

where

$$\dot{\mathcal{H}}_\mathrm{e}(x_t, \dot{x}_t, u) = \mathcal{L}_\mathcal{F}^2\mathcal{H}(x_t, \dot{x}_t) + \mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}\mathcal{H}(x_t)\,u + \alpha'\big(\mathcal{H}(x_t)\big)\mathcal{L}_\mathcal{F}\mathcal{H}(x_t), \tag{61}$$

is the derivative of $\mathcal{H}_\mathrm{e}$ along (44), given by $\dot{x}_t$ in (45) and the functionals $\mathcal{L}_\mathcal{F}^2\mathcal{H} : \mathcal{B} \times \mathcal{Q} \to \mathbb{R}$ and $\mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}\mathcal{H} : \mathcal{B} \to \mathbb{R}^{1\times m}$.

Note that condition (60) sufficiently holds if $\mathcal{L}_\mathcal{G}\mathcal{L}_\mathcal{F}\mathcal{H}(x_t) \neq 0, \forall x_t \in \mathcal{B}$, i.e., in case $\mathcal{H}$ has relative degree 2. With this definition we can state the theorem to ensure safety for systems with (potentially delay-induced) relative degree 2.

Theorem 6. *If* $\mathcal{H}_\mathrm{e}$ *is an extended CBFal for* (44) *on* $\mathcal{S} \cap \mathcal{S}_\mathrm{e}$ *defined by* (20) *and* (59)*, then any locally Lipschitz continuous controller* $\mathcal{K} : \mathcal{B} \times \mathcal{Q} \to \mathbb{R}^m$, $u = \mathcal{K}(x_t, \dot{x}_t)$ *satisfying:*

$$\dot{\mathcal{H}}_\mathrm{e}(x_t, \dot{x}_t, \mathcal{K}(x_t, \dot{x}_t)) \geq -\alpha_\mathrm{e}\big(\mathcal{H}_\mathrm{e}(x_t)\big), \tag{62}$$

$\forall x_t \in \mathcal{S} \cap \mathcal{S}_\mathrm{e}$ *renders set* $\mathcal{S} \cap \mathcal{S}_\mathrm{e}$ *forward invariant w.r.t.* (46).

Proof. We prove that $x_0 \in \mathcal{S} \cap \mathcal{S}_\mathrm{e} \implies x_t \in \mathcal{S} \cap \mathcal{S}_\mathrm{e}$, $\forall t \geq 0$. Note that $x_0 \in \mathcal{S} \cap \mathcal{S}_\mathrm{e}$ yields both $x_0 \in \mathcal{S}$ and $x_0 \in \mathcal{S}_\mathrm{e}$. By Theorem 5, we have $x_0 \in \mathcal{S}_\mathrm{e} \implies x_t \in \mathcal{S}_\mathrm{e}$, $\forall t \geq 0$. This means $\mathcal{H}_\mathrm{e}(x_t) \geq 0$ holds, i.e., $\dot{\mathcal{H}}(x_t) \geq -\alpha\big(\mathcal{H}(x_t)\big)$ based on (58). Therefore, applying Theorem 3 (or more precisely, Corollary 5) yields $x_0 \in \mathcal{S} \cap \mathcal{S}_\mathrm{e} \implies x_t \in \mathcal{S} \cap \mathcal{S}_\mathrm{e}$, $\forall t \geq 0$. □

With this theorem, one can design a pointwise optimal controller similarly as in (52).

Corollary 6. *Given an extended CBFal $\mathcal{H}_{\rm e}$ and a locally Lipschitz continuous desired controller $\mathcal{K}_{\rm des}: \mathcal{B}\times\mathcal{Q}\to\mathbb{R}^m$, $u_{\rm des}=\mathcal{K}_{\rm des}(x_t,\dot{x}_t)$, the following quadratic program (QP) yields a controller $\mathcal{K}: \mathcal{B}\times\mathcal{Q}\to\mathbb{R}^m$, $u=\mathcal{K}(x_t,\dot{x}_t)$ that renders set $\mathcal{S}\cap\mathcal{S}_{\rm e}$ in* (20) *and* (59) *forward invariant w.r.t.* (46)*:*

$$\begin{aligned}\mathcal{K}(x_t,\dot{x}_t)=\underset{u\in\mathbb{R}^m}{\operatorname{argmin}}\quad & \frac{1}{2}\|u-\mathcal{K}_{\rm des}(x_t,\dot{x}_t)\|_2^2\\ \text{s.t.}\quad & \dot{\mathcal{H}}_{\rm e}(x_t,\dot{x}_t,u)\geq-\alpha_{\rm e}\big(\mathcal{H}_{\rm e}(x_t)\big).\end{aligned}\tag{63}$$

Furthermore, the explicit solution of (63) *can be found by the Karush–Kuhn–Tucker (KKT)*[60] *conditions as:*

$$\mathcal{K}(x_t,\dot{x}_t)=\begin{cases}\mathcal{K}_{\rm des}(x_t,\dot{x}_t) & \text{if } \phi_{\rm e}(x_t,\dot{x}_t)\geq 0,\\ \mathcal{K}_{\rm des}(x_t,\dot{x}_t)-\frac{\phi_{\rm e}(x_t,\dot{x}_t)\phi_{0\rm e}^\top(x_t)}{\phi_{0\rm e}(x_t)\phi_{0\rm e}^\top(x_t)} & \text{otherwise,}\end{cases}\tag{64}$$

where $\phi_{\rm e}(x_t,\dot{x}_t)=\mathcal{L}_{\mathcal{F}}^2\mathcal{H}(x_t,\dot{x}_t)+\mathcal{L}_{\mathcal{G}}\mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t)\,\mathcal{K}_{\rm des}(x_t,\dot{x}_t)+\alpha'\big(\mathcal{H}(x_t)\big)\mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t)+\alpha_{\rm e}\,\big(\mathcal{H}_{\rm e}(x_t)\big)$ and $\phi_{0\rm e}(x_t)=\mathcal{L}_{\mathcal{G}}\mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t)$.

The detailed derivation of (64) is similar to (52) which is discussed in Appendix D.

5 | EXAMPLES AND APPLICATION

Now we apply the theoretical constructions of this paper on demonstrative examples and a relevant practical application.

5.1 | Illustrative Examples

First, we address systems with point delay, second, we discuss a scalar control system with different types of delay.

Example 2. Based on the motivating example (2)-(3), consider the following affine control system with point delay $\tau>0$:

$$\dot{x}(t)=f\big(x(t),x(t-\tau)\big)+g\big(x(t),x(t-\tau)\big)u(t),\tag{65}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, while $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are locally Lipschitz continuous. To keep this system safe, we seek to enforce:

$$h\big(x(t),x(t-\tau)\big)\geq 0,\quad \forall t\geq 0,\tag{66}$$

along the solutions of the corresponding closed control loop, where $h:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is continuously differentiable.

System (65) and the corresponding safe set can be rewritten in the form as (44) and (20) with the functionals $\mathcal{F}:\mathcal{B}\to\mathbb{R}^n$, $\mathcal{G}:\mathcal{B}\to\mathbb{R}^{n\times m}$ and $\mathcal{H}:\mathcal{B}\to\mathbb{R}$ defined by:

$$\begin{aligned}\mathcal{F}(x_t)&=f\big(x_t(0),x_t(-\tau)\big),\\ \mathcal{G}(x_t)&=g\big(x_t(0),x_t(-\tau)\big),\\ \mathcal{H}(x_t)&=h\big(x_t(0),x_t(-\tau)\big).\end{aligned}\tag{67}$$

This choice of functional $\mathcal{H}$ is a special case of that in (30) in Example 1. The time derivative of $\mathcal{H}$ becomes:

$$\dot{\mathcal{H}}(x_t,\dot{x}_t,u)=\nabla_0 h\big(x_t(0),x_t(-\tau)\big)\Big(f\big(x_t(0),x_t(-\tau)\big)+g\big(x_t(0),x_t(-\tau)\big)u(t)\Big)+\nabla_1 h\big(x_t(0),x_t(-\tau)\big)\dot{x}_t(-\tau),\tag{68}$$

cf. (31), where ∇_j denotes the gradient with respect to the j-th vector-valued variable. This expression corresponds to (48) with:

$$\begin{aligned}\mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t,\dot{x}_t)=&\nabla_0 h\big(x_t(0),x_t(-\tau)\big)f\big(x_t(0),x_t(-\tau)\big)+\nabla_1 h\big(x_t(0),x_t(-\tau)\big)\dot{x}_t(-\tau),\\ \mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t)=&\nabla_0 h\big(x_t(0),x_t(-\tau)\big)g\big(x_t(0),x_t(-\tau)\big),\end{aligned}\tag{69}$$

and the weights $w_0(x_t)=\nabla_0 h\big(x_t(0),x_t(-\tau)\big)$, $w_1(x_t)=\nabla_1 h\big(x_t(0),x_t(-\tau)\big)$ and $w_{\rm d}(x_t,\vartheta)=0$ in (49). Thereafter, safety-critical controllers can be designed based on (50) in Theorem 5, such the quadratic program-based controller (52).

Example 3. Consider the scalar control system:

$$\dot{x}(t)=\underbrace{x^3(t)}_{\mathcal{F}(x_t)}+\underbrace{x(t-\tau)}_{\mathcal{G}(x_t)}u(t),\tag{70}$$

where $x \in \mathbb{R}$, $u \in \mathbb{R}$. Notice that this system is not forward complete and for some controllers (including $u(t) \equiv 0$) the solution may have finite escape time. Hence, we seek to keep the state within safe bounds. We compare different types of CBFals, that include delay-free, point delay and distributed delay terms, as special cases of the functional in (30) in Example 1.

Case 1: First, we intend to keep the solution $x(t)$ within the safe range of $[-1,\ 1]$, defined by $-1 \le x(t) \le 1$. To enforce this, we construct the CBFal with delay-free term as:

$$\mathcal{H}(x_t) = 1 - x^2(t), \tag{71}$$

i.e., $\mathcal{H}(x_t) \ge 0$ yields safety. This CBFal corresponds to (30) with $h(s, \dots) = 1 - s^2$, and the time derivative of the CBFal reads:

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, u) = -2x(t)\left(x^3(t) + x(t-\tau)u\right) \tag{72}$$

with weights $w_0(x_t) = -2x(t)$, $w_j(x_t) = 0$, $w_{\rm d}(x_t, \vartheta) = 0$, while the expressions in (49) become:

$$\begin{aligned} \mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t, \dot{x}_t) &= -\,2x^4(t), \\ \mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t) &= -\,2x(t)x(t-\tau). \end{aligned} \tag{73}$$

We apply Theorem 5 to ensure safety and implement the QP-based controller (52) using the desired controller $\mathcal{K}_{\rm des}(x_t, \dot{x}_t) = 0$ and a linear class-$\mathcal{K}_\infty^{\rm e}$ function $\alpha(r) = \gamma r$ with $\gamma > 0$, that results in:

$$\mathcal{K}(x_t, \dot{x}_t) = \begin{cases} 0 & \text{if } -2x^4(t) + \gamma\left(1 - x^2(t)\right) \ge 0, \\ \frac{-2x^4(t) + \gamma\left(1 - x^2(t)\right)}{2x(t)x(t-\tau)} & \text{otherwise.} \end{cases} \tag{74}$$

Substitution back into (70) leads to the closed-loop system that forms the differential equation:

$$\dot{x}(t) = \begin{cases} x^3(t) & \text{if } -2x^4(t) + \gamma\left(1 - x^2(t)\right) \ge 0, \\ \frac{\gamma\left(1 - x^2(t)\right)}{2x(t)} & \text{otherwise.} \end{cases} \tag{75}$$

It is important that although in this example the delayed term dropped from the closed-loop system, this is not the case in general, and it depends on the forms of (70) and (71).

The performance of this controller is demonstrated by numerically integrating (75). Simulation results are plotted in the first column of Fig. 3 (panels (a),(d),(g),(j)) for $\tau = 1$, $\gamma = 1$ and initial condition $x_0(\vartheta) = 0.4$, $\vartheta \in [-\tau, 0]$. The panel (a) shows the evolution of the state, and the panel (d) indicates the corresponding control input. As the state gets close to the safe set boundary (indicated by green line), the controller needs to intervene by deviating from the desired zero input (see around $t = 2$) and forces the system to evolve within the safe set. Intervention starts when the state reaches the switching surface in (75): $-2x^4(t) + \gamma\left(1 - x^2(t)\right) = 0$ (see the dashed line in panel (a)). Panels (g), (j) at the bottom indicate that safety is successfully maintained as $\mathcal{H}$ is positive for all time while the trajectory in the corresponding phase portrait is kept within $-1 \le x(t) \le 1$.

Case 2: Next, we intend to keep the squared mean of the solution $x(t)$ and its delayed value $x(t-\tau)$ below 1, defined by $(x^2(t) + x^2(t-\tau))/2 \le 1$. Thus we construct the CBFal as:

$$\mathcal{H}(x_t) = 1 - \frac{1}{2}\left(x^2(t) + x^2(t-\tau)\right), \tag{76}$$

corresponding to the nonlinear function h in (30) with $h(s_1, s_2, \dots) = 1 - \frac{1}{2}(s_1^2 + s_2^2)$. The derivative of the CBFal becomes:

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, u) = -x(t)\left(x^3(t) + x(t-\tau)u\right) - x(t-\tau)\dot{x}(t-\tau), \tag{77}$$

associated with weights $w_0(x_t) = -x(t)$, $w_j(x_t) = -x(t-\tau)$, $w_{\rm d}(x_t, \vartheta) = 0$, leading to:

$$\begin{aligned} \mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t, \dot{x}_t) &= -\,x^4(t) - x(t-\tau)\dot{x}(t-\tau), \\ \mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t) &= -\,x(t)x(t-\tau), \end{aligned} \tag{78}$$

cf. (49).

Implementing the QP-based controller (52) with $\mathcal{K}_{\rm des}(x_t, \dot{x}_t) = 0$ and $\alpha(r) = \gamma r$ results in:

$$\mathcal{K}(x_t, \dot{x}_t) = \begin{cases} 0 & \text{if } -x^4(t) - x(t-\tau)\dot{x}(t-\tau) + \gamma\left(1 - \frac{1}{2}\left(x^2(t) + x^2(t-\tau)\right)\right) \ge 0, \\ \frac{-x^4(t) - x(t-\tau)\dot{x}(t-\tau) + \gamma\left(1 - \frac{1}{2}\left(x^2(t) + x^2(t-\tau)\right)\right)}{x(t)x(t-\tau)} & \text{otherwise,} \end{cases} \tag{79}$$

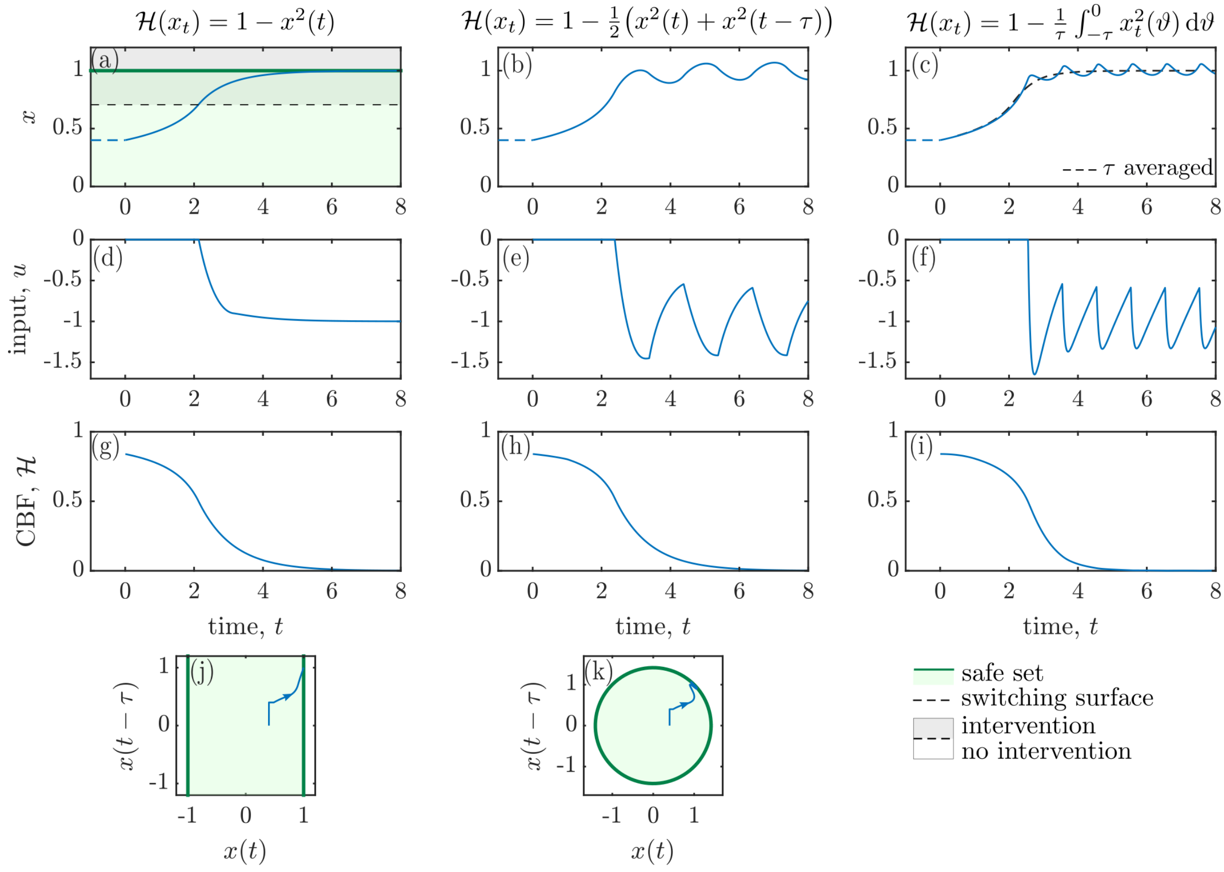

FIGURE 3 Safety-critical control of system (70) with different types of CBFals. The left column corresponds to *Case 1* with delay-free CBFal (71), the middle column shows the results of *Case 2* with point delay in the CBFal (76), while the right column represents *Case 3* with distributed delay in the functional (81). Panels (a), (b), (c) show the evolution of the state for the three different scenarios, panels (d), (e), (f) plot the value of the synthesized control inputs that maintain safety, panels (g), (h), (i) depict the value of the corresponding functional $\mathcal{H}$, while panels (j), (k) show state space plots.

while the closed-loop system becomes a neutral delay differential equation:

$$\dot{x}(t) = \begin{cases} x^3(t) & \text{if } -x^4(t) - x(t-\tau)\dot{x}(t-\tau) + \gamma\left(1 - \frac{1}{2}\left(x^2(t) + x^2(t-\tau)\right)\right) \geq 0, \\ \frac{-x(t-\tau)\dot{x}(t-\tau) + \gamma\left(1-\frac{1}{2}\left(x^2(t)+x^2(t-\tau)\right)\right)}{x(t)} & \text{otherwise.} \end{cases} \tag{80}$$

The time derivative of the delayed term, $\dot{x}(t-\tau)$, appears in the expressions of both the right-hand side and the switching surface.

The middle column of Fig. 3 (panels (b),(e),(h),(k)) plots simulation results for system (80) with $\tau = 1$, $\gamma = 1$ and initial conditions $x_0(\vartheta) = 0.4$, $\vartheta \in [-\tau, 0]$ and $\dot{x}_0(\vartheta) = 0$, $\vartheta \in [-\tau, 0)$. Again, the safety-critical controller keeps the system within the safe set for all time as guaranteed by Theorem 5. This can be clearly seen from the phase portrait in panel (k). Note that in steady state the intervention is not constant (as it was in Case 1) but periodic, and the system approaches a periodic solution.

Case 3: Then, we intend to keep a moving average of the solution (i.e., a root-mean-square average over the delay interval) below 1, defined by $\int_{-\tau}^{0} x^2(t+\vartheta)/\tau \,\mathrm{d}\vartheta \leq 1$. The CBFal candidate that can enforce this is defined with distributed delay as:

$$\mathcal{H}(x_t) = 1 - \frac{1}{\tau}\int_{-\tau}^{0} x^2(t+\vartheta)\,\mathrm{d}\vartheta. \tag{81}$$

This is a special case of (30) with $h(\dots, s) = s$, $\rho(\vartheta) \equiv \frac{1}{\tau}$ and $\kappa(s) = 1 - s^2$. Taking the time derivative of $\mathcal{H}$ and simplifying the integral as in Remark 2, we get:

$$\dot{\mathcal{H}}(x_t) = \frac{1}{\tau}\big(x^2(t-\tau) - x^2(t)\big), \tag{82}$$

associated with:

$$\begin{aligned} L_{\mathcal{F}}\mathcal{H}(x_t) &= \frac{1}{\tau}\big(x^2(t-\tau) - x^2(t)\big), \\ L_{\mathcal{G}}\mathcal{H}(x_t) &= 0. \end{aligned} \tag{83}$$

Observe that $\dot{\mathcal{H}}$ depends only on x_t, as emphasized by the notation $\dot{\mathcal{H}}(x_t)$. Therefore, functional $\mathcal{H}$ in (81) is not a valid CBFal because $L_{\mathcal{G}}\mathcal{H}(x_t) = 0$ and u does not appear in $\dot{\mathcal{H}}$. However, since $\dot{x}_t$ does not appear in $\dot{\mathcal{H}}$ either, one may construct the extended CBFal in Definition 9. Based on (58), we propose the following extended CBFal:

$$\mathcal{H}_{\rm e}(x_t) = \frac{1}{\tau}\big(x^2(t-\tau) - x^2(t)\big) + \gamma\left(1 - \frac{1}{\tau}\int_{-\tau}^{0} x^2(t+\vartheta)\,\mathrm{d}\vartheta\right), \tag{84}$$

whose time derivative is:

$$\dot{\mathcal{H}}_{\rm e}(x_t, \dot{x}_t, u) = \frac{2}{\tau}x(t-\tau)\dot{x}(t-\tau) - \frac{2}{\tau}x^4(t) - \frac{2}{\tau}x(t)x(t-\tau)u + \frac{\gamma}{\tau}\big(x^2(t-\tau) - x^2(t)\big), \tag{85}$$

cf. (61). This depends on the control input u, and it is associated with:

$$\begin{aligned} L_{\mathcal{F}}^2\mathcal{H}(x_t, \dot{x}_t) &= -\frac{2}{\tau}x^4(t) + \frac{2}{\tau}x(t-\tau)\dot{x}(t-\tau), \\ L_{\mathcal{G}}L_{\mathcal{F}}\mathcal{H}(x_t) &= -\frac{2}{\tau}x(t)x(t-\tau). \end{aligned} \tag{86}$$

Now we use Theorem 6 to guarantee safety, and consider a min-norm controller by setting $\mathcal{K}_{\rm des}(x_t, \dot{x}_t) = 0$, choosing $\alpha_{\rm e}(r) = \gamma_{\rm e} r$ with $\gamma_{\rm e} > 0$ and using (64). This yields the controller:

$$\mathcal{K}(x_t, \dot{x}_t) = \begin{cases} 0 & \text{if } \dfrac{-2x^4(t)+2x(t-\tau)\dot{x}(t-\tau)+(\gamma+\gamma_{\rm e})\big(x^2(t-\tau)-x^2(t)\big)+\gamma_{\rm e}\gamma\big(\tau-\int_{-\tau}^{0} x_t^2(\vartheta)\,\mathrm{d}\vartheta\big)}{\tau} \geq 0, \\ \dfrac{-2x^4(t)+2x(t-\tau)\dot{x}(t-\tau)+(\gamma+\gamma_{\rm e})\big(x^2(t-\tau)-x^2(t)\big)+\gamma_{\rm e}\gamma\big(\tau-\int_{-\tau}^{0} x_t^2(\vartheta)\,\mathrm{d}\vartheta\big)}{2x(t)x(t-\tau)} & \text{otherwise,} \end{cases} \tag{87}$$

and the closed-loop system:

$$\dot{x}(t) = \begin{cases} x^3(t) & \text{if } \dfrac{-2x^4(t)+2x(t-\tau)\dot{x}(t-\tau)+(\gamma+\gamma_{\rm e})\big(x^2(t-\tau)-x^2(t)\big)+\gamma_{\rm e}\gamma\big(\tau-\int_{-\tau}^{0} x_t^2(\vartheta)\,\mathrm{d}\vartheta\big)}{\tau} \geq 0, \\ \dfrac{2x(t-\tau)\dot{x}(t-\tau)+(\gamma+\gamma_{\rm e})\big(x^2(t-\tau)-x^2(t)\big)+\gamma_{\rm e}\gamma\big(\tau-\int_{-\tau}^{0} x_t^2(\vartheta)\,\mathrm{d}\vartheta\big)}{2x(t)} & \text{otherwise,} \end{cases} \tag{88}$$

which is an integro-differential equation with neutral-type delay term.

Simulation results are presented in the right column of Fig. 3 (panels (c),(f),(i)) for system (88) with $\tau = 1$, $\gamma_{\rm e} = 1$, $\gamma = 3$ and initial conditions $x_0(\vartheta) = 0.4$, $\vartheta \in [-\tau, 0]$ and $\dot{x}_0(\vartheta) = 0$, $\vartheta \in [-\tau, 0)$. Although the state variable is sometimes greater than one, the controller forces the moving average of the solution (dashed curve) to evolve within the safe set for all time, as desired.

To summarize Cases 1-3, if the functional $\mathcal{H}$ depends only on the delay-free state, then the QP-based control law depends only on the state x_t and the closed-loop system is an RFDE. If there is delay in the functional $\mathcal{H}$, then the control law may depend not only on the state x_t, but also on its time derivative $\dot{x}_t$, leading to a nonsmooth NFDE as the closed-loop system.

Case 4: Finally, we show an example of a functional that is not a valid CBFal and does not have a valid relative degree due to delays. Consider the following functional constructed by the combination of point delay and distributed delay terms:

$$\tilde{\mathcal{H}}(x_t) = 1 + \frac{1}{2}x^2(t-\tau) - \frac{1}{\tau}\int_{-\tau}^{0} x^2(t+\vartheta)\,\mathrm{d}\vartheta. \tag{89}$$

Note that $\tilde{\mathcal{H}}$ does not include the present state, only the past. Here we use tilde to emphasize that this functional does not satisfy condition I in Definition 8 and hence does not have a valid relative degree. Taking its time derivative leads to:

$$\dot{\tilde{\mathcal{H}}}(x_t, \dot{x}_t) = x(t-\tau)\dot{x}(t-\tau) + \frac{1}{\tau}\big(x^2(t-\tau) - x^2(t)\big), \tag{90}$$

associated with:

$$\mathcal{L}_F\tilde{\mathcal{H}}(x_t,\dot{x}_t) = x(t-\tau)\dot{x}(t-\tau) + \frac{1}{\tau}\big(x^2(t-\tau) - x^2(t)\big),$$
$$\mathcal{L}_G\tilde{\mathcal{H}}(x_t) = 0, \tag{91}$$

cf. (49). One can observe that $\mathcal{L}_G\tilde{\mathcal{H}}(x_t) = 0$ similar to Case 3. Thus, one may construct an extended functional according to (58):

$$\tilde{\mathcal{H}}_{\rm e}(x_t,\dot{x}_t) = x(t-\tau)\dot{x}(t-\tau) + \frac{1}{\tau}\big(x^2(t-\tau) - x^2(t)\big) + \gamma\left(1 + \frac{1}{2}x^2(t-\tau) - \frac{1}{\tau}\int_{-\tau}^{0} x^2(t+\vartheta)\,{\rm d}\vartheta\right), \tag{92}$$

however, this functional depends on $\dot{x}_t$ as opposed to the extended CBFal in Definition 9. Taking its time derivative gives:

$$\dot{\tilde{\mathcal{H}}}_{\rm e}(x_t,\dot{x}_t,\ddot{x}_t,u) = x(t-\tau)\ddot{x}(t-\tau)+\dot{x}^2(t-\tau)+\frac{2}{\tau}x(t-\tau)\dot{x}(t-\tau)-\frac{2}{\tau}x^4(t)-\frac{2}{\tau}x(t)x(t-\tau)u+\gamma x(t-\tau)\dot{x}(t-\tau)+\frac{\gamma}{\tau}\big(x^2(t-\tau)-x^2(t)\big), \tag{93}$$

in which the advanced-type term $\ddot{x}(t-\tau)$ appears, as emphasized by the notation $\dot{\tilde{\mathcal{H}}}_{\rm e}(x_t,\dot{x}_t,\ddot{x}_t,u)$. Synthesizing a QP-based safe controller from this based on (63) (e.g. with desired controller $\mathcal{K}_{\rm des}(x_t,\dot{x}_t) = 0$ and linear class-$\mathcal{K}^{\rm e}_\infty$ function) would result in a control law $u = \mathcal{K}(x_t,\dot{x}_t,\ddot{x}_t)$ that depends on $\ddot{x}_t$. This means that the corresponding closed-loop dynamics would become an advanced functional differential equation with inverted causality problem and no well-defined solutions. Therefore, the functional in (89) cannot be used for safety-critical control. Intuitively, (89) represents the requirement of keeping the past of the system safe without considering the present—this is not feasible, as captured by the fact that there is no valid relative degree.

5.2 | Case-study: Regulated Delayed Predator-prey Model

Finally, we investigate a predator-prey problem which is subject to time delay [76,25,77]. This application becomes more and more important considering the fragile ecosystems as a consequence of climate change. The evolution of predator and prey populations is described in an ecosystem, and human intervention, regarded as control input, is used to control these numbers [78]. We use the proposed safety-critical control framework to regulate the numbers of predators or preys and maintain these numbers within safe bounds. Naturally, such ecosystems contain significant time delays because each predator and prey has a finite maturation period in its life before they start interacting with each other. While the underlying delayed dynamics have been analysed extensively during the last few decades by many researchers [79], to the best of our knowledge, safety-critical control has not yet been applied to address this problem due to the lack of theoretical background that endows time delay systems with provable guarantees of safety. Now we use this problem to demonstrate that our proposed framework is able to provide the desired safety guarantees for systems with state delays.

Description	Parameter	Value
prey growth rate	r	1
prey self-regulation rate	a	1
predation rate of the prey	p	4
conversion rate of prey into predator	b	1.2
predator intraspecific competition	m	0.1
predator mortality rate	d	1
predator maturation time	τ	5
prey population lower limit	$x_{1,\min}$	0.05
prey population upper limit	$x_{1,\max}$	0.6

TABLE 1 Parameters of the predator-prey model.

We denote the population of the preys by x_1 and the population of the predators by x_2, and we model their dynamic interaction by the following nondimensionalized system:

$$\underbrace{\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} rx_1(t) - ax_1^2(t) - px_1(t)x_2(t) \\ bpx_1(t-\tau)x_2(t-\tau) - dx_2(t) - mx_2^2(t)\end{bmatrix}}_{f(x(t),x(t-\tau))} + \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{g(x(t))} u(t), \tag{94}$$

where the time delay τ indicates the maturation time of the predators, r describes the growth rate of the prey in the absence of predators, a denotes the self-regulation constant of the prey, p describes the predation rate of the prey by predators, b indicates the rate of conversion of consumed prey to predator, d is the specific mortality of predator in the absence of prey, m describes the intraspecific competition among predators [80], and u quantifies the effect of human intervention affecting the number of predators.

Without control input ($u(t) \equiv 0$), system (94) has four equilibria: $(x_1, x_2) = (0, 0)$, $(x_1, x_2) = (0, -d/m)$, $(x_1, x_2) = (r/a, 0)$ and $(x_1, x_2) = (mr + pd, bpr - ad)/(am + bp^2)$, from which the last one is relevant. It corresponds to a constant population in the ecosystem, which is independent of the time delay. However, its stability depends on the delay, and there exists a critical time delay above which the equilibrium becomes unstable by a supercritical Hopf bifurcation that induces stable periodic solutions [77]. In this case, from the biological point of view, the populations of predators and preys are oscillating.

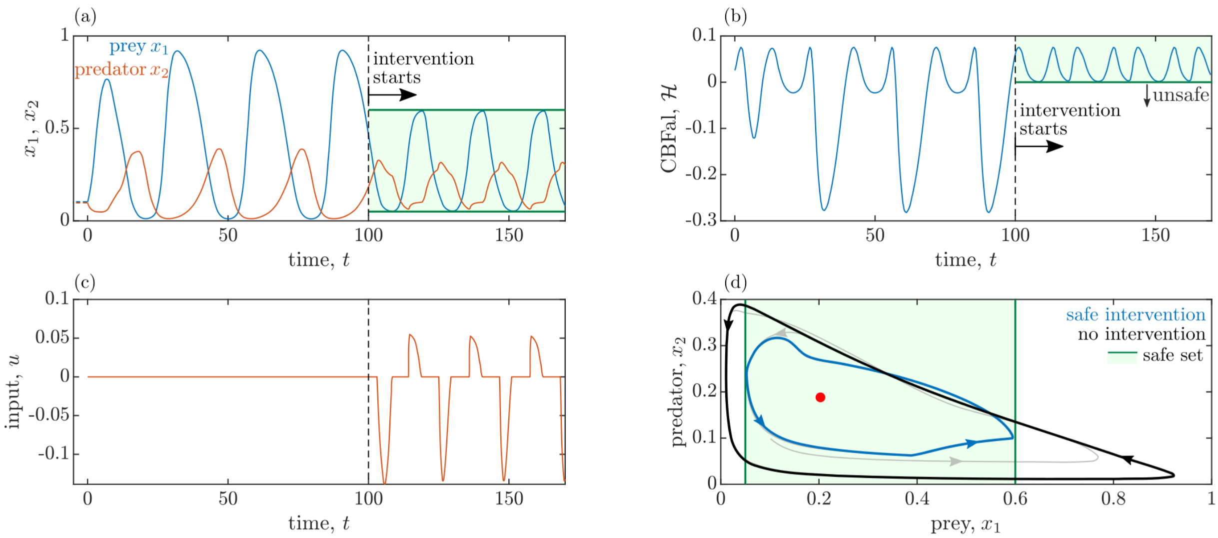

FIGURE 4 Safety-critical control of the delayed predator-prey model (94) to keep the number of preys within safe limits. While the ecosystem is unsafe without intervention, the proposed control strategy, that relies on control barrier functionals, is able regulate the ecosystem in a provably safe fashion.

In the literature there exist several methods to regulate the predator-prey system such as the addition of food, pesticide or insecticide [81,78]. In this example, we directly control the number of predators by capturing or releasing them, i.e., the control input u enters the dynamics of the predator population x_2.

We seek to synthesize a controller that keeps the ecosystem safe. Specifically, our aim is to regulate the number x_1 of preys and keep their population within prescribed bounds. On one hand, we specify a lower limit $x_{1,\min}$ to avoid getting close to the danger of prey extinction. On the other hand, we include an upper limit $x_{1,\max}$, to prevent the number of preys from increasing too much, which could result in an unsustainable ecosystem with over-consumption of available food resources. These requirements can be formulated as $x_{1,\min} \leq x_1(t) \leq x_{1,\max}$.

Thus, we use the candidate control barrier functional:

$$\mathcal{H}(x_t) = -\big(x_1(t) - x_{1,\min}\big)\big(x_1(t) - x_{1,\max}\big), \tag{95}$$

and we seek to maintain $\mathcal{H}(x_t) \geq 0$ for all time. The time derivative of $\mathcal{H}$ along (94) reads:

$$\dot{\mathcal{H}}(x_t) = 2\left(\bar{x}_1 - x_1(t)\right)\left(rx_1(t) - ax_1^2(t) - px_1(t)x_2(t)\right) \tag{96}$$

with $\bar{x}_1 = (x_{1,\min} + x_{1,\max})/2$. Since $\dot{\mathcal{H}}$ only depends on x_t (i.e., the corresponding $\mathcal{L}_{\mathcal{F}}\mathcal{H}$ does not contain terms of $\dot{x}_t$ and $\mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t) = 0$), we construct the extended CBFal (58) with $\alpha(r) = \gamma r$, $\gamma > 0$, that becomes:

$$\mathcal{H}_{\mathrm{e}}(x_t) = 2\left(\bar{x}_1 - x_1(t)\right)\left(rx_1(t) - ax_1^2(t) - px_1(t)x_2(t)\right) - \gamma(x_1(t) - x_{1,\min})(x_1(t) - x_{1,\max}). \tag{97}$$

Its derivative,

$$\dot{\mathcal{H}}_{\mathrm{e}}(x_t, \dot{x}_t, u) = -2\left(rx_1(t) - ax_1^2(t) - px_1(t)x_2(t)\right)^2 + 2\left(\bar{x}_1 - x_1(t)\right)\left(r - 2ax_1(t) - px_2(t)\right)\left(rx_1(t) - ax_1^2(t) - px_1(t)x_2(t)\right)$$
$$- 2p\left(\bar{x}_1 - x_1(t)\right)x_1(t)\left(bpx_1(t-\tau)x_2(t-\tau) - dx_2(t) - mx_2^2(t) + u\right) + 2\gamma\left(\bar{x}_1 - x_1(t)\right)\left(rx_1(t) - ax_1^2(t) - px_1(t)x_2(t)\right), \tag{98}$$

depends on the input u. Thereby safety can be enforced by synthesizing controllers according to Theorem 6. We implement the QP-based controller (64), with desired controller $\mathcal{K}_{\mathrm{des}}(x_t, \dot{x}_t) = 0$ and linear class-$\mathcal{K}_\infty^{\mathrm{e}}$ function $\alpha_{\mathrm{e}}(r) = \gamma_{\mathrm{e}} r$, $\gamma_{\mathrm{e}} > 0$.

Simulation results are illustrated in Fig. 4 for the model parameters are listed in Table 1, $\gamma = 1$, $\gamma_{\mathrm{e}} = 1$ and initial conditions $x_t(\vartheta) = \begin{bmatrix}0.1 & 0.1\end{bmatrix}^\top$, $\vartheta \in [-\tau, 0]$. First, we demonstrate the natural evolution of system (94) without active intervention ($u(t) \equiv 0$) during the time interval $t \in [0, 100]$. Then, the safety-critical controller is turned on after $t = 100$.

Without safety-critical control ($t < 100$), the trajectory of the system converges to a stable periodic orbit with large fluctuation in the populations (see black orbit in panel (d)). Along this periodic orbit, the number of prey increases when there are a few predators. Afterwards, the growing number of prey provides sufficient food for predators so their population also increases. Once the predator population exceeds a certain level, the population of the prey starts to decrease due to overconsumption. Then, due to lack of food, the predator population also starts shrinking. This cycle of growing and shrinking populations repeats periodically. Notice that during this process the prey population goes outside the range $[x_{1,\min}, x_{1,\max}]$, and the system periodically leaves the safe set as indicated by the negative values of the functional $\mathcal{H}$ during $t \in [0, 100)$.

With safety-critical control ($t \geq 100$), the level of intervention is quantified in panel (c). To keep the number of preys within the prescribed limits, predators are captured ($u(t) < 0$) when the population of preys is too small and predators are added ($u(t) > 0$) when the prey population is too large. The controller intervenes minimally in the system, only when the prey population is too close to the safe limits, and otherwise lets the predator and prey populations evolve naturally (i.e., $u(t) = 0$ for some duration of time). The effect of this intervention on safety is clearly seen: it maintains nonnegative values for the functional $\mathcal{H}$, keeps the number of preys within safe bounds, and forces the system to evolve within the safe set (green shaded areas in panels (a) and (d)). In fact, the solution converges to a stable limit cycle (blue orbit in panel (d)) located inside the safe set (and grazing the boundaries of the safe set periodically). Compared to the uncontrolled scenario, the proposed controller modifies the size of the stable limit cycle around the unstable equilibrium without changing these stability properties. Remarkably, such desired behavior is generated by a systematic controller synthesis procedure based on control barrier functional theory.

6 | CONCLUSION

In this work, we have discussed the safety of time delay systems that include state delays. First, we have proposed a method to formally certify the safety of autonomous delayed dynamical systems by means of *safety functionals* defined over the infinite dimensional state space. We have broken down how the required time derivative of the potentially complicated nonlinear safety functional can be expressed. Second, we have extended this theory to control systems with time delay in order synthesize safety-critical controllers that are endowed with rigorous safety guarantees. The essence of the proposed method is the introduction of *control barrier functionals* that guide the selection of safe control inputs for time delay systems in a similar fashion to how control barrier functions yield safety for delay-free systems. We have provided formal safety guarantees with their proofs. Third, we have also incorporated control barrier functionals into optimization problems to find pointwise optimal safe controllers, and we have demonstrated the applicability of the proposed method on multiple examples.

The proposed theoretical framework open ways towards provably safe and reliable operation of time delay systems. While the paper presents some demonstrative examples, we believe that this approach could be used in a much wider range of application fields. Our future goals are to implement this method in engineering applications and to perform experiments to validate the theoretical results.

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Author contributions

A. K. Kiss developed the theory and conducted the numerical simulations. T. G. Molnar supported the theory and helped with the writing. A. D. Ames and G. Orosz supervised the project, including the development of the theory, results and writing.

Conflict of interest

The authors declare no potential conflict of interests.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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APPENDIX

A PROOF OF THEOREM 1

Proof of Theorem 1. The proof is given by the comparison lemma[82], and for further technical nuances, please refer to[59]. To set up the comparison lemma, consider the scalar initial value problem (with $y \in \mathbb{R}$):

$$\dot{y}(t) = -\alpha\big(y(t)\big), \qquad y(0) = h(x(0)), \tag{A1}$$

with the following solution (note that (A1) has a unique solution because α is an extended class-$\mathcal{K}_\infty$ function[59]):

$$y(t) = \beta\big(h(x(0)), t\big), \tag{A2}$$

for $t \geq 0$, where $\beta \in \mathcal{K}^{\mathrm{e}}_\infty\mathcal{L}$ (see footnote[3]). By the comparison lemma for (6) and (A1), we obtain:

$$h(x(t)) \geq \beta\big(h(x(0)), t\big), \quad \forall t \geq 0. \tag{A3}$$

Therefore, $h(x(0)) \geq 0 \implies h(x(t)) \geq 0, \forall t \geq 0$, that is, S is forward invariant. This completes the proof. □

[3]Function $\beta : \mathbb{R} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is of extended class-$\mathcal{K}_\infty\mathcal{L}$, denoted as $\beta \in \mathcal{K}^{\mathrm{e}}_\infty\mathcal{L}$, if $\beta(.,s) \in \mathcal{K}^{\mathrm{e}}_\infty$ for any $s \in \mathbb{R}_{\geq 0}$, and $|\beta(r,.)|$ is decreasing and $\lim_{s\to\infty} \beta(r,s) = 0$ for any $r \in R$.

B DERIVATIVES OF FUNCTIONALS

This appendix shows the necessary definitions and lemma for the proof of Theorem 4.

Recall that for the delay-free system in Section 2, the time derivative of function h along the system can be considered as the directional derivative of h along $\dot{x}$. This is generalized to functionals in that the time derivative of functional $\mathcal{H}$ is the directional derivative of $\mathcal{H}$ along $\dot{x}_t$, which is given by the so-called Gâteaux derivative formulated as follows.

Definition 10 (Gâteaux Derivative). Functional $\mathcal{H}: \mathcal{B} \to \mathbb{R}$ is Gâteaux differentiable at $x_t \in \mathcal{B}$ if there exists a functional $D_{\mathrm{G}}\mathcal{H}(x_t): \mathcal{Q} \to \mathbb{R}$ such that $\forall \phi \in \mathcal{Q}$:

$$D_{\mathrm{G}}\mathcal{H}(x_t)(\phi) = \lim_{\Delta t \to 0} \frac{\mathcal{H}(x_t + \Delta t \phi) - \mathcal{H}(x_t)}{\Delta t}, \quad \Delta t \in \mathbb{R}, \tag{B4}$$

where $D_{\mathrm{G}}\mathcal{H}(x_t)$ is called the **Gâteaux derivative** of $\mathcal{H}$ at x_t, that is evaluated along ϕ.

Similarly, the generalization of the gradient ∇h is the so-called Fréchet derivative of $\mathcal{H}$, defined by the implicit form below.

Definition 11 (Fréchet derivative). Functional $\mathcal{H}: \mathcal{B} \to \mathbb{R}$ is Fréchet differentiable at $x_t \in \mathcal{B}$ if there exists a bounded linear functional $D_{\mathrm{F}}\mathcal{H}(x_t): \mathcal{Q} \to \mathbb{R}$ such that:

$$\lim_{\|\phi\| \to 0} \frac{|\mathcal{H}(x_t + \phi) - \mathcal{H}(x_t) - D_{\mathrm{F}}\mathcal{H}(x_t)\phi|}{\|\phi\|} = 0, \quad \phi \in \mathcal{Q}, \tag{B5}$$

where $D_{\mathrm{F}}\mathcal{H}(x_t)$ is called the **Fréchet derivative** of $\mathcal{H}$ at x_t.

If $\mathcal{H}$ is Fréchet differentiable, it implies that it is also Gâteaux differentiable and directional derivatives exist in all directions. The following lemma formally establishes this connection between the Gâteaux and Fréchet derivatives to prove Theorem 4.

Lemma 1. [83] *If $\mathcal{H}: \mathcal{B} \to \mathbb{R}$ is Fréchet differentiable at $x_t \in \mathcal{B}$, then it is also Gâteaux differentiable at x_t, and the Gâteaux derivative $D_{\mathrm{G}}\mathcal{H}(x_t): \mathcal{Q} \to \mathbb{R}$ is given by a bounded linear functional that is the Fréchet derivative $D_{\mathrm{F}}\mathcal{H}(x_t): \mathcal{Q} \to \mathbb{R}$:*

$$D_{\mathrm{G}}\mathcal{H}(x_t)(\dot{x}_t) = D_{\mathrm{F}}\mathcal{H}(x_t)\dot{x}_t, \quad \forall \dot{x}_t \in \mathcal{Q}. \tag{B6}$$

The proof can be found in[83], Proposition A.3. Consequently, if $\mathcal{H}$ is continuously Fréchet differentiable, its Fréchet and Gâteaux derivatives are continuous linear functionals. While nonlinear functionals have no general form, continuous linear functionals can be represented in the form of Stieltjes integrals, as provided by the Riesz representation theorem.

Theorem 7 (Riesz Representation Theorem[84,44,47,85,45]). *For every continuous linear functional $\mathcal{L}: \mathcal{B} \to \mathbb{R}$ there exists a unique function $\eta: [-\tau, 0] \to \mathbb{R}^{1\times n}$ that is of bounded variation such that $\forall \phi \in \mathcal{B}$:*

$$\mathcal{L}(\phi) = \int_{-\tau}^{0} \mathrm{d}_\vartheta \eta(\vartheta)\phi(\vartheta), \tag{B7}$$

where the integral is a Stieltjes type and $\mathrm{d}_\vartheta\eta(\vartheta)$ is interpreted as a measure corresponding to the function η.

The proof can be found in[84]. Accordingly, the Fréchet derivative of a continuously Fréchet differentiable functional $\mathcal{H}: \mathcal{B} \to \mathbb{R}$ is a continuous linear functional, that can be represented as follows when evaluated along $\dot{x}_t$:

$$D_{\mathrm{F}}\mathcal{H}(x_t)\dot{x}_t = \int_{-\tau}^{0} \mathrm{d}_\vartheta \eta(x_t, \vartheta)\dot{x}_t(\vartheta). \tag{B8}$$

Theorem 4 is therefore a direct consequence of Lemma 1 and Theorem 7.

C EXAMPLE FUNCTIONAL WITH DOUBLE INTEGRAL

Let us consider the system (17) with the functional $\mathcal{H}$ defined as:

$$\mathcal{H}(x_t) = h\Bigg(x_t(0), \int_{-\tau}^{0} \rho(\vartheta)\, \kappa\big(x_t(\vartheta)\big)\mathrm{d}\vartheta, \int_{-\tau}^{0}\int_{-\tau}^{0} \omega(\vartheta,\chi)\, \Big(\mu\big(x_t(\vartheta)\big) \circ \nu\big(x_t(\chi)\big)\Big)\mathrm{d}\vartheta\mathrm{d}\chi \Bigg), \tag{C9}$$

cf. (30), where $h:\ \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $\kappa,\mu,\nu:\ \mathbb{R}^n\to\mathbb{R}^n$ are nonlinear functions, $\circ$ refers to element-wise multiplication, while the density functions $\rho:\ [-\tau,0]\to\mathbb{R}^{n\times n}$ and $\omega:\ [-\tau,0]\times[-\tau,0]\to\mathbb{R}^{n\times n}$ are assumed to be continuously differentiable and these functions allow us take into account the states between time moments $t-\tau$ and t.

One may take the time derivative of $\mathcal{H}$ in (C9), substitute (19), and use

$$\begin{aligned}\frac{\partial}{\partial t}\kappa\big(x_t(\vartheta)\big)&=\frac{\partial}{\partial\vartheta}\kappa\big(x_t(\vartheta)\big),\\ \frac{\partial}{\partial t}\mu\big(x_t(\vartheta)\big)&=\frac{\partial}{\partial\vartheta}\mu\big(x_t(\vartheta)\big),\\ \frac{\partial}{\partial t}\nu\big(x_t(\chi)\big)&=\frac{\partial}{\partial\chi}\nu\big(x_t(\chi)\big).\end{aligned}\tag{C10}$$

By executing partial integration, these steps yield:

$$\begin{aligned}\dot{\mathcal{H}}(x_t,\dot x_t)=&\ \nabla_1 h(\dots)\mathcal{F}(x_t)\\ &+\nabla_2 h(\dots)\bigg(\rho(0)\kappa\big(x(t)\big)-\rho(-\tau)\kappa\big(x(t-\tau)\big)-\int_{-\tau}^{0}\rho'(\vartheta)\,\kappa\big(x(t+\vartheta)\big)\mathrm{d}\vartheta\bigg)\\ &+\nabla_3 h(\dots)\bigg(\int_{-\tau}^{0}\omega(0,\chi)\Big(\mu\big(x(t)\big)\circ\nu\big(x(t+\chi)\big)\Big)-\omega(-\tau,\chi)\Big(\mu\big(x(t-\tau)\big)\circ\nu\big(x(t+\chi)\big)\Big)\mathrm{d}\chi\\ &\quad+\int_{-\tau}^{0}\omega(\vartheta,0)\Big(\mu\big(x(t+\vartheta)\big)\circ\nu\big(x(t)\big)\Big)-\omega(\vartheta,-\tau)\Big(\mu\big(x(t+\vartheta)\big)\circ\nu\big(x(t-\tau)\big)\Big)\mathrm{d}\vartheta\\ &\quad-\int_{-\tau}^{0}\int_{-\tau}^{0}\Big(\frac{\partial}{\partial\vartheta}\omega(\vartheta,\chi)+\frac{\partial}{\partial\vartheta}\omega(\vartheta,\chi)\Big)\Big(\mu\big(x(t+\vartheta)\big)\circ\nu\big(x(t+\chi)\big)\Big)\mathrm{d}\vartheta\mathrm{d}\chi\bigg),\end{aligned}\tag{C11}$$

cf. (31), where ∇_j represents the gradient with respect to the j-th vector-valued variable, and $(\dots)$ is a shorthand notation for evaluation at the argument of h as in (C9). Note that terms of $\dot x_t$ drop from $\dot{\mathcal{H}}$ similar to the case in Remark 2.

D KARUSH-KUHN-TUCKER CONDITIONS

Here, we briefly discuss the derivation steps for determining the solution (52) to the quadratic program (51) in Corollary 4.

Let us define $\Delta\mathcal{K}:\ \mathcal{B}\times\mathcal{Q}\to\mathbb{R}^m$, $\Delta\mathcal{K}(x_t,\dot x_t)=\mathcal{K}(x_t,\dot x_t)-\mathcal{K}_{\rm des}(x_t,\dot x_t)$ and consider the expressions of $\dot{\mathcal{H}}$ in (48) and ϕ, ϕ_0 in Corollary 4. Then, we can restate (51) as:

$$\begin{aligned}\mathcal{K}(x_t,\dot x_t)&=\mathcal{K}_{\rm des}(x_t,\dot x_t)+\Delta\mathcal{K}(x_t,\dot x_t),\\ \Delta\mathcal{K}(x_t,\dot x_t)&=\underset{\Delta u\in\mathbb{R}^m}{\operatorname{argmin}}\quad\frac{1}{2}\|\Delta u\|_2^2\\ &\qquad\text{s.t.}\quad \phi(x_t,\dot x_t)+\phi_0(x_t,\dot x_t)\Delta u\geq 0.\end{aligned}\tag{D12}$$

In order to solve (D12), let us define the *Lagrangian* $L:\ \mathbb{R}^m\times\mathcal{B}\times\mathcal{Q}\to\mathbb{R}$ associated with the optimization problem (D12) as:

$$L(\Delta u,x_t,\dot x_t)=\|\Delta u(x_t,\dot x_t)\|_2^2-\mu(x_t,\dot x_t)\big(\phi(x_t,\dot x_t)+\phi_0(x_t,\dot x_t)\Delta u\big),\tag{D13}$$

where $\mu:\ \mathcal{B}\times\mathcal{Q}\to\mathbb{R}$ is the Lagrange multiplier associated with the inequality constraint. This optimization problem has convex objective and affine constraint, hence the *Karush-Kuhn-Tucker (KKT) conditions*[60] provide the necessary and sufficient conditions for optimality, listed as:

$$\mu(x_t,\dot x_t)\geq 0,\qquad\text{Dual Feasibility}\tag{D14}$$

$$\Delta\mathcal{K}(x_t,\dot x_t)=\mu(x_t,\dot x_t)\phi_0^\top(x_t,\dot x_t),\qquad\text{Stationary}\tag{D15}$$

$$\phi(x_t,\dot x_t)+\phi_0(x_t,\dot x_t)\Delta\mathcal{K}(x_t,\dot x_t)\geq 0,\qquad\text{Primal Feasibility}\tag{D16}$$

$$\mu(x_t,\dot x_t)\big(\phi(x_t,\dot x_t)+\phi_0(x_t,\dot x_t)\Delta\mathcal{K}(x_t,\dot x_t)\big)=0.\qquad\text{Complementary Slackness}\tag{D17}$$

We decompose the dual feasibility condition (D14) into two cases: $\mu(x_t, \dot{x}_t) = 0$ and $\mu(x_t, \dot{x}_t) > 0$. If $\mu(x_t, \dot{x}_t) = 0$, then the stationary condition (D15) leads to:

$$\Delta\mathcal{K}(x_t, \dot{x}_t) = 0, \tag{D18}$$

while the primal feasibility condition (D16) implies $\phi(x_t, \dot{x}_t) \geq 0$. If $\mu(x_t, \dot{x}_t) > 0$, then the complementary slackness condition (D17) yields:

$$\phi(x_t, \dot{x}_t) + \phi_0(x_t, \dot{x}_t)\Delta\mathcal{K}(x_t, \dot{x}_t) = 0, \tag{D19}$$

and we can express $\Delta\mathcal{K}$ as:

$$\Delta\mathcal{K}(x_t, \dot{x}_t) = -\frac{\phi(x_t, \dot{x}_t)\phi_0^\top(x_t, \dot{x}_t)}{\phi_0(x_t, \dot{x}_t)\phi_0^\top(x_t, \dot{x}_t)}. \tag{D20}$$

With the stationary condition (D15) this also means $\mu(x_t, \dot{x}_t) = -\phi(x_t, \dot{x}_t)/(\phi_0(x_t, \dot{x}_t)\phi_0^\top(x_t, \dot{x}_t))$ that implies $\phi(x_t, \dot{x}_t) < 0$ since $\mu(x_t, \dot{x}_t) > 0$.

Then the closed-form solution of the quadratic program (51) can be written as:

$$\mathcal{K}(x_t, \dot{x}_t) = \begin{cases} \mathcal{K}_{\text{des}}(x_t, \dot{x}_t) & \text{if } \phi(x_t, \dot{x}_t) \geq 0, \\ \mathcal{K}_{\text{des}}(x_t, \dot{x}_t) - \frac{\phi(x_t, \dot{x}_t)\phi_0^\top(x_t, \dot{x}_t)}{\phi_0(x_t, \dot{x}_t)\phi_0^\top(x_t, \dot{x}_t)} & \text{otherwise,} \end{cases} \tag{D21}$$

cf. (52). Note that the CBFal condition (47) is equivalent to $\mathcal{L}_\mathcal{G}\mathcal{H}(x_t, \dot{x}_t) = 0 \implies \mathcal{L}_\mathcal{F}\mathcal{H}(x_t, \dot{x}_t) + \alpha\big(\mathcal{H}(x_t)\big) > 0$, and we have $\phi_0(x_t, \dot{x}_t) = 0 \implies \phi(x_t, \dot{x}_t) > 0$. Hence division by zero cannot occur in (D21) if $\mathcal{H}$ is a CBFal.